A Gentle Introduction to Lyapunov Functions

Maya Ramakrishnan
ORSUM 19th August 2003

The University of Melbourne
Outline

1. Motivation
2. Basic Definitions
3. Methods of determining stability
   - Indirect Method
   - Direct Method
4. Example
5. Applications
6. Conclusion
Motivation

Often systems can be represented or modelled using systems of differential equations. Given a system, questions which naturally arise include:

- What are the equilibrium points or orbits?
- Is the system stable?
- What are the dominant features of the portrait of the orbit?
Basic Definitions

Consider the system

\[ \dot{x} = f(x), \quad (1) \]

where \( f \) is not necessarily a linear function of \( x \).

**Definition 1** If \( x \) is an equilibrium point of the system, then \( f(x) = 0 \).
A system is stable at an equilibrium point if it responds to small changes from the equilibrium with only small changes in its subsequent states.

**Definition 2** An equilibrium point, \( \bar{x} \), is said to be stable if for all \( \epsilon > 0 \), there exists \( \delta = \delta(\epsilon) \), such that

\[
|x(t_0) - \bar{x}| < \delta \Rightarrow |x(t) - \bar{x}| < \epsilon, \quad \forall t > t_0. \tag{2}
\]

Note. A system can be *locally* stable.
Basic Definitions cont’d

**Definition 3**  
We say that $\bar{x}$ is asymptotically stable if it is stable and

$$|x(t) - \bar{x}| \to 0, \text{ as } t \to \infty.$$  \hfill (3)

**Definition 4**  
A system is neutrally stable at $\bar{x}$ if it is stable at $\bar{x}$, but not asymptotically stable.

**Definition 5**  
A system that is not stable at $\bar{x}$ is said to be unstable.
Stablility of $\bar{x}$

1. Lyapunov’s first (indirect) method

Consider the system above, where $f$ is not a linear function. Using Taylor expansion, we have

$$\dot{x} = Ax + g(x), \quad (4)$$

where $A$ is the Jacobian matrix of $f(x)$ evaluated at $\bar{x}$, ie.

$$A = \frac{\partial f}{\partial x} \bigg|_{\bar{x}}, \quad (5)$$

and $g(x)$ is a function of the higher order terms.
Stability of $\bar{x}$: Method 1

The resulting linear system is given by

$$\dot{x} = Ax. \quad (6)$$

The stability of the system can then be assessed using the following rules:

1. The equilibrium point is asymptotically stable if the linearised system is strictly stable
2. The equilibrium point is unstable if the linearised system is strictly unstable
3. If the linearised system is marginally stable, then we cannot conclude anything from the linear approximation.
Stability of $\bar{x}$: Method 1

Consider 1. from the previous slide. If all the eigenvalues of $A$ have negative real parts, then the linearised system is asymptotically stable.

Problems include:

- If some of the eigenvalues are zero, we cannot conclude anything about the system.
- The linear approximation is valid when the initial conditions are close to $\bar{x}$, however how "close" is close?
Stability of \( \bar{x} \): Method 2

Lyapunov’s Second (or direct) Method

Stability is determined by observing how a scalar function of the state variables of the system changes in time.

Consider the system

\[
\dot{x} = f(x),
\]

(7)

where

- \( \bar{x} \) is the equilibrium point of the system;
- \( f \in C^1(R) \), where \( C^k(R) \) defines the class of all continuous functions on region \( R \) with \( k \)th derivatives also continuous on \( R \); and
- \( x(t^0) = x^0, \ x^0 \in R \).
Stability of $\bar{x}$: Method 2

Definition 6  Let $V(x)$ be a real valued scalar function belonging to $C^0(S)$ for some region $S$ of $\mathbb{R}^n$. Suppose $V(\bar{x}) = 0$. Then,

1. $V$ is positive definite on $S$ if $V(x) > 0$, for $x \neq \bar{x}$

2. $V$ is negative definite on $S$ if $V(x) < 0$, for $x \neq \bar{x}$

3. If the equalities are not strict, then $V$ is considered positive and negative semi-definite respectively
Stability of \( \bar{x} \): Method 2

**Definition 7** Let \( V(x) \) be a real valued scalar function belonging to \( C^1(R) \). The total derivative along the solution curves of \( \dot{x} = f(x) \), i.e. with respect to the orbits \( x = x(t, x^0) \) is

\[
\dot{V} = \frac{d}{dt} V(x(t, x^0)) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i(t, x^0) = \nabla V(x)^T f(x).
\]
**Theorem 1** Suppose there exists a function $V(x)$ with the following properties:

1. $V(\bar{x}) = 0$
2. $V(x)$ is positive definite for $x \neq \bar{x}$
3. $\dot{V}(x)$ is negative semi-definite along trajectories of $\dot{x} = f(x)$.

Then $V(x)$ is known as a Lyapunov function and $\bar{x}$ is stable.

Note. If $\dot{V}(x)$ is negative definite, then $\bar{x}$ is asymptotically stable.
**Proof (sufficiency)**

**Proof 1** This proof will assume that the equilibrium point $\bar{x} = 0$ for simplicity. It is easily extendable to the situation where this is not the case.

Let

- $J_\epsilon$ be the interior of a sphere of radius $\epsilon$, whose centre is at point 0. Further, assume $\epsilon$ is such that $J_\epsilon$ lies in the region $R$.
- $S_\epsilon$ be the spherical surface of the sphere.

Let $V$ be a function that satisfies properties 1 - 3 from the previous slide.
Proof cont’d

Since $V(x)$ is continuous and positive on the closed, bounded surface $S_\varepsilon$, by the Minimum Value Theorem for continuous functions, $V(x)$ has a minimum on $S_\varepsilon$. Suppose this minimum is $L$. Then:

$$0 < L \leq V(x), \quad \forall x \in S_\varepsilon. \quad (8)$$

Since $V(x)$ is continuous and has value 0 at $x = 0$, there exists an open ball $J_\delta$ centred at 0, which lies entirely inside $J_\varepsilon$.

- Choose $\delta$ such that $V(x) < L$, $\forall x \in J_\delta$
Proof cont’d

Let $x^0$ be a point inside $J_\delta$.

- Consider the trajectory $x(x^0, t)$ coming from point $x^0$ and satisfying the system of differential equations under consideration.

- Assume the trajectory travels outside the the sphere $S_\epsilon$.
  - Since the trajectory is continuous, it will intersect the boundary $S_\epsilon$ at some point $x^*$
Proof cont’d

- Recall that $V(x)$ satisfies the property

$$\dot{V} = \sum_{i=0}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i \leq 0$$ \hspace{1cm} (9)

Hence, the function $V$ is not increasing along the trajectory of $x(x^0, t)$.

- This means that

$$V(x^*) \leq V(x^0) < L.$$ \hspace{1cm} (10)
Proof cont’d

But $x^* \in S_\epsilon$ and therefore $L \leq V(x^*)$. This yields a contradiction. Hence as $t$ increases, the trajectory $x(x^0, t)$ does not pass outside the limits of $S_\epsilon$.

We have shown, with $\bar{x} = 0$, that for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, such that

$$|x^0 - \bar{x}| < \delta \Rightarrow |x(x^0, t) - \bar{x}| < \epsilon$$

as required.
Example

Consider the system:

\[
\begin{align*}
\dot{x}_1 &= -x_2 + ax_1x_2^2 \\
\dot{x}_2 &= x_1 - bx_1^2x_2,
\end{align*}
\]

where \(a \neq b\).

To find the equilibrium points, set

\[
\begin{align*}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= 0.
\end{align*}
\]
Example cont’d

Basic manipulation shows that the equilibrium point is \( \bar{x}_1 = \bar{x}_2 = 0 \).

Question: Is this equilibrium point asymptotically stable?

**Indirect Method**

Consider the linearised system:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = Ax
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]
Example cont’d

Next consider the eigenvalues, $\lambda$, of $A$.

\[
det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0. \tag{11}
\]

- Recall for asymptotic stability of the linear system, we require all eigenvalues to have negative real parts
- The roots of this equation are purely imaginary
- Hence we cannot draw any conclusions about the stability of the non-linear system
Example cont’d

Direct Method

Consider the function

\[ V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \]  

(12)

Clearly,

- \( V(\bar{x}) = 0 \)
- When \( x \neq \bar{x} \), \( V(x) > 0 \), ie. it is positive definite.
Example cont’d

Next consider $\dot{V}(x)$:

$$
\dot{V}(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2 \\
= x_1(-x_2 + ax_1^2) + x_2(x_1 - bx_1^2 x_2) \\
= x_1 x_2^2 (a - b).
$$

For asymptotic stability, we need $\dot{V}(x)$ to be negative definite. Hence, if $a < b$, then the system is asymptotically stable.
Applications

Some common applications of Lyapunov functions are in the area of:

- Assessing the importance of non-linear terms in stability and instability
- Estimating the domain of attraction of an equilibrium point
- Designing control laws that guarantee global asymptotic stability
Applications

Rate control for communication networks, *Kelly et al (1998)*

- Proposed optimization framework
  - Larger system problem decomposed into a User and a Network problem

- Proposed rate control algorithm
  - System of differential equations in the rate/capacity allocated to each user

- With the correct choice of Lyapunov function, it is shown that:
  - The found equilibrium point is asymptotically stable
  - The Lyapunov function closely approximates the Network’s optimisation problem
Conclusion

- Lyapunov functions are useful in assessing the stability of systems and in particular the method can be used:
  - For exploring non-linear systems
  - For time varying systems, ie $\dot{x} = f(x, t)$
  - To determine both stability and asymptotic stability
- Drawback is that to find a Lyapunov function is often more of an art than a science