The Paradox of Parrondo’s Games

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Parrondo’s Games

Game A

This is a straight-forward toss of a weighted coin.The probabilities are

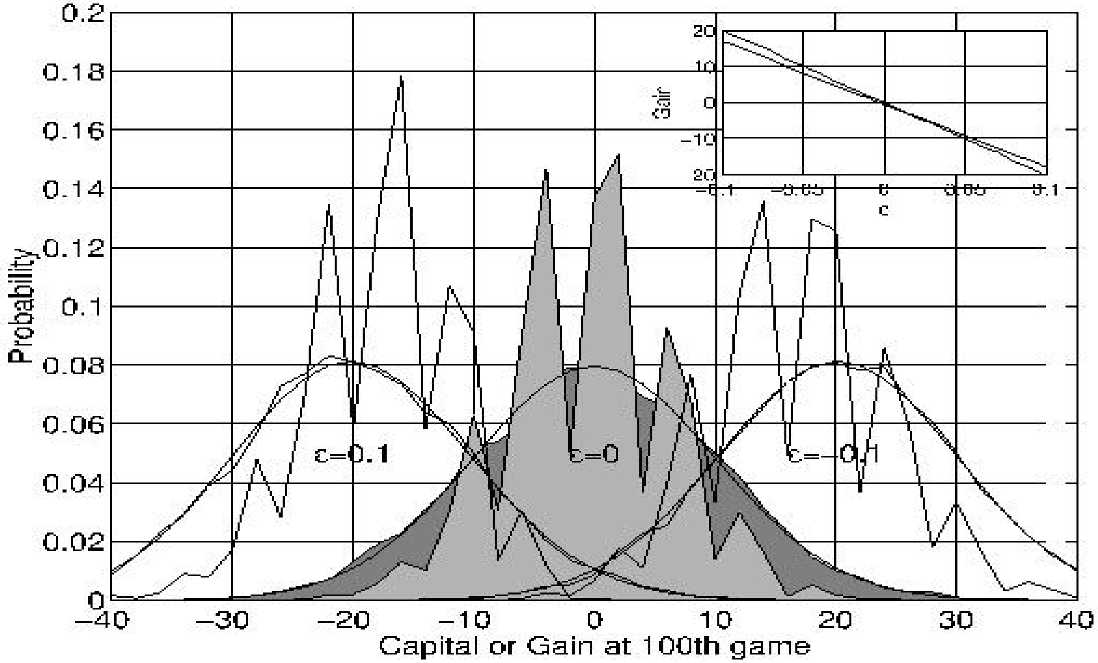
$$\begin{aligned} P[\text{win}] &= p \\ P[\text{lose}] &= 1 - p. \end{aligned}$$

Game B

If the player's present capital is a multiple of M then the chance of winning is p_1, if it is not a multiple of M the chance of winning is p_2.

Let j be the player's current capital. Then

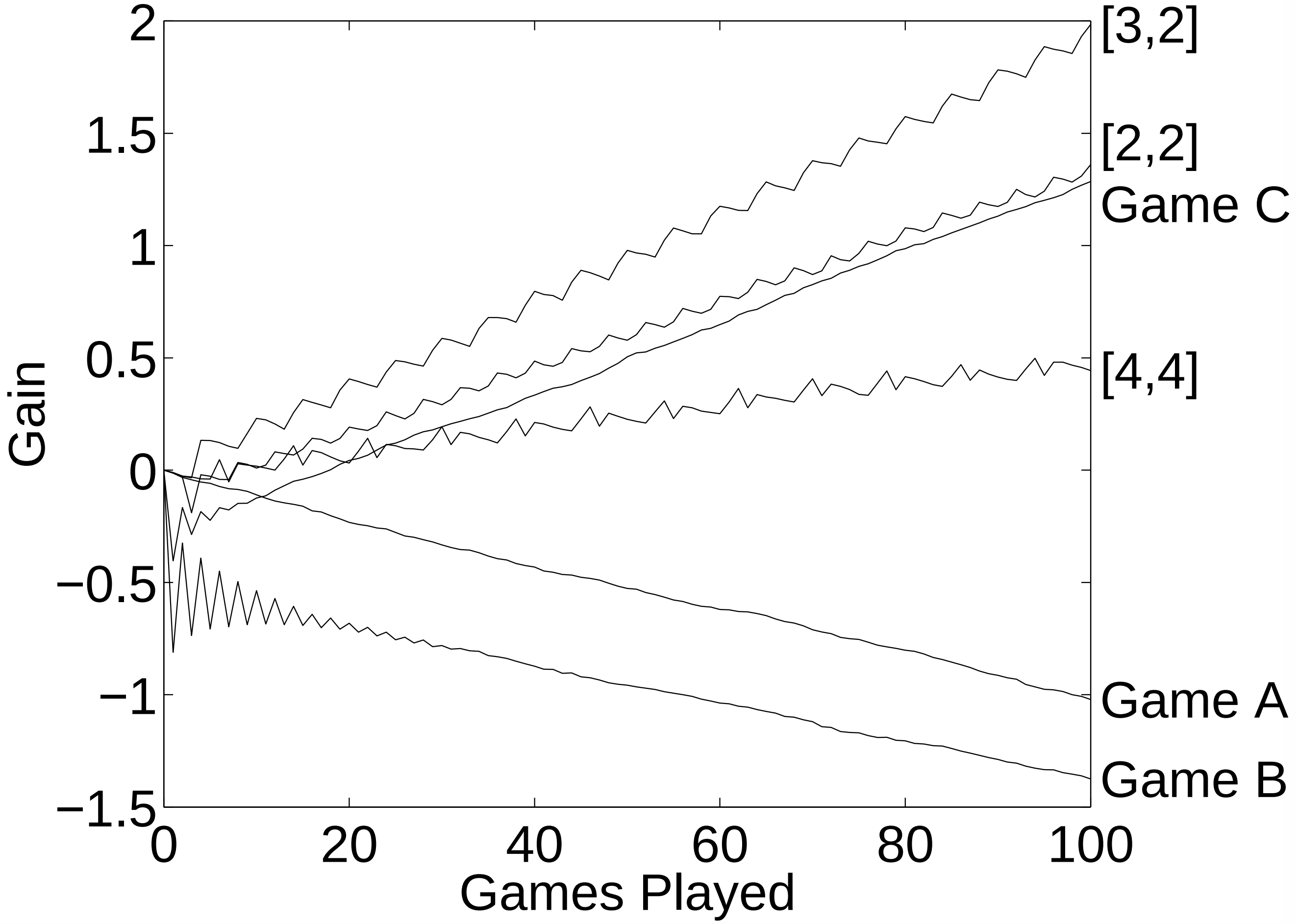
$$\begin{aligned}
P[\text{win}|j = 0(mod\ M)] &= p_1, \\
P[\text{lose}|j = 0(mod\ M)] &= 1 - p_1, \\
P[\text{win}|j \neq 0(mod\ M)] &= p_2, \\
P[\text{lose}|j \neq 0(mod\ M)] &= 1 - p_2.
\end{aligned}$$



Probability
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
−40
−30
−20
−10
0
10
20
30
40
Capital or Gain at 100th game
ε=0.1
ε=0
ε=−0.1
Gain
20
10
0
−10
20
−0.1
−0.05
0
0.05
0.1
ε

A third game, which we shall call game C, can be constructed by choosing randomly between games A and B. Thus

- In game C, game A is chosen with probability γ and game B with probability $(1 - \gamma)$.



2
1.5
1
0.5
0
−0.5
−1
−1.5
Gain
0
20
40
60
80
100
Games Played
[3,2]
[2,2]
Game C
[4,4]
Game A
Game B

There appear to be choices of p, p_1, p_2 and γ such that games A and B are both losing, but game C is winning.

This behaviour, which at first sight seems surprising, has been termed Parrondo's paradox.

The Definition of Fairness

Consider the situation in which a gambler repeatedly plays a game and where the gambler's fortune after the nth game is X_n. In Doob (1953) it was stated that the game is fair if

$$\mathbf{E}(X_{n+1}|X_0, X_1, \ldots X_n) = X_n.$$

In other words, a fair game is a martingale.

Corresponding definitions can be used to state that winning game is a submartingale and a losing game is a supermartingale.

Note, however, that Doob observed that the above definition

is somewhat arbitrary, although hallowed by tradition.

Using Doob’s definition, when $p_1 < 1/2$ and $p_2 > 1/2$, game B defined above is neither a fair, winning or losing game because, when X_0 is a multiple of M, $\mathbf{E}(X_1|X_0) < X_0$ and, when X_0 is not a multiple of M, $\mathbf{E}(X_1|X_0) > X_0$.

Thus the stochastic process $\{X_n\}$ is neither a martingale, a supermartingale nor a submartingale.

Consider a game whose capital moves up and down by one unit at a time and whose winning probabilities depend just on the current capital.

We propose the following definition. Such a game is

- fair if its corresponding birth and death process is null recurrent,
- losing if its corresponding birth and death process is positive recurrent,
- winning if its corresponding birth and death process is transient.

Analysis of the Games.

For Game A, the transition probabilities of the birth and death process are

$$p_{ij} = \begin{cases} p & \text{if } j = i+1, \\ 1-p & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

For Game B, the transition probabilities are

$$\begin{aligned} p_{i,i+1} &= \begin{cases} p_1 & \text{if } i = 0(mod\ M), \\ p_2 & \text{if } i \neq 0(mod\ M), \end{cases} \\ p_{i,i-1} &= \begin{cases} 1 - p_1 & \text{if } i = 0(mod\ M), \\ 1 - p_2 & \text{if } i \neq 0(mod\ M), \end{cases} \\ p_{ij} &= 0 \text{ otherwise.} \end{aligned}$$

We need to decide whether the birth and death processes are recurrent. One way to do this is to calculate the probability f_j that process visits 0 given that it starts in state j.

This is known to be the minimal non-negative solution to

$$f_j = p_{j,j+1} f_{j+1} + p_{j,j-1} f_{j-1}$$

with

$$f_0 = 1.$$

For Game A, these equations become

$$f_j = pf_{j+1} + (1-p)f_{j-1}, \quad j \geq 1,$$

with

$$f_0 = 1.$$

The minimal nonnegative solution to this is given by

$$f_j = \min\left(\left(\frac{1-p}{p}\right)^j, 1\right).$$

Thus the game is

1. winning if $\left(\frac{1-p}{p}\right) < 1$, that is if $p > 1/2$,
2. losing if $\left(\frac{1-p}{p}\right) > 1$, that is if $p < 1/2$ and
3. fair if $\left(\frac{1-p}{p}\right) = 1$, that is if $p = 1/2$.

This result, of course, accords with our intuition.

Using similar analysis as for Game A we deduce that Game B is winning if

$$\left(\frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}}\right) < 1,$$

losing if

$$\left(\frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}}\right) > 1$$

and fair if

$$\left(\frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}}\right) = 1.$$

Note that the parameters $M = 3$, $p_1 = 1/10$ and $p_2 = 3/4$ satisfy the third equation.

Parrondo’s Paradox.

What about Parrondo’s Paradox?

Assume the player plays Game A with probability γ and Game B with probability $1-\gamma$.

If their capital is a multiple of M the probability that the player wins the randomized game is

$$q_1 = \gamma p + (1-\gamma)p_1.$$

If their capital is not a multiple of M the probability that the player wins the randomized game is

$$q_2 = \gamma p + (1-\gamma)p_2.$$

The probabilities of losing are $1-q_1$ and $1-q_2$ respectively.

Observe that this is identical to Game B except that the probabilities have changed. It follows that the randomised game is winning if

$$\left(\frac{(1-q_1)(1-q_2)^{M-1}}{q_1 q_2^{M-1}}\right) < 1,$$

losing if

$$\left(\frac{(1-q_1)(1-q_2)^{M-1}}{q_1 q_2^{M-1}}\right) > 1$$

and fair if

$$\left(\frac{(1-q_1)(1-q_2)^{M-1}}{q_1 q_2^{M-1}}\right) = 1.$$

The existence of Parrondo’s paradox will be established if we can find parameters M, p, p_1, p_2 and γ for which

$$\left(\frac{1-p}{p}\right) > 1,$$

$$\left(\frac{(1-p_1)(1-p_2)^{M-1}}{p_1 p_2^{M-1}}\right) > 1$$

and

$$\left(\frac{(1-q_1)(1-q_2)^{M-1}}{q_1 q_2^{M-1}}\right) < 1.$$

If we take $M = 3$, $p = 5/11$, $p_1 = 1/121$, $p_2 = 10/11$ and $\gamma = 1/2$, then

$$\left(\frac{1-p}{p}\right) = 6/5 > 1,$$

$$\left(\frac{(1-p_1)(1-p_2)^2}{p_1 p_2^2}\right) = 6/5 > 1,$$

but

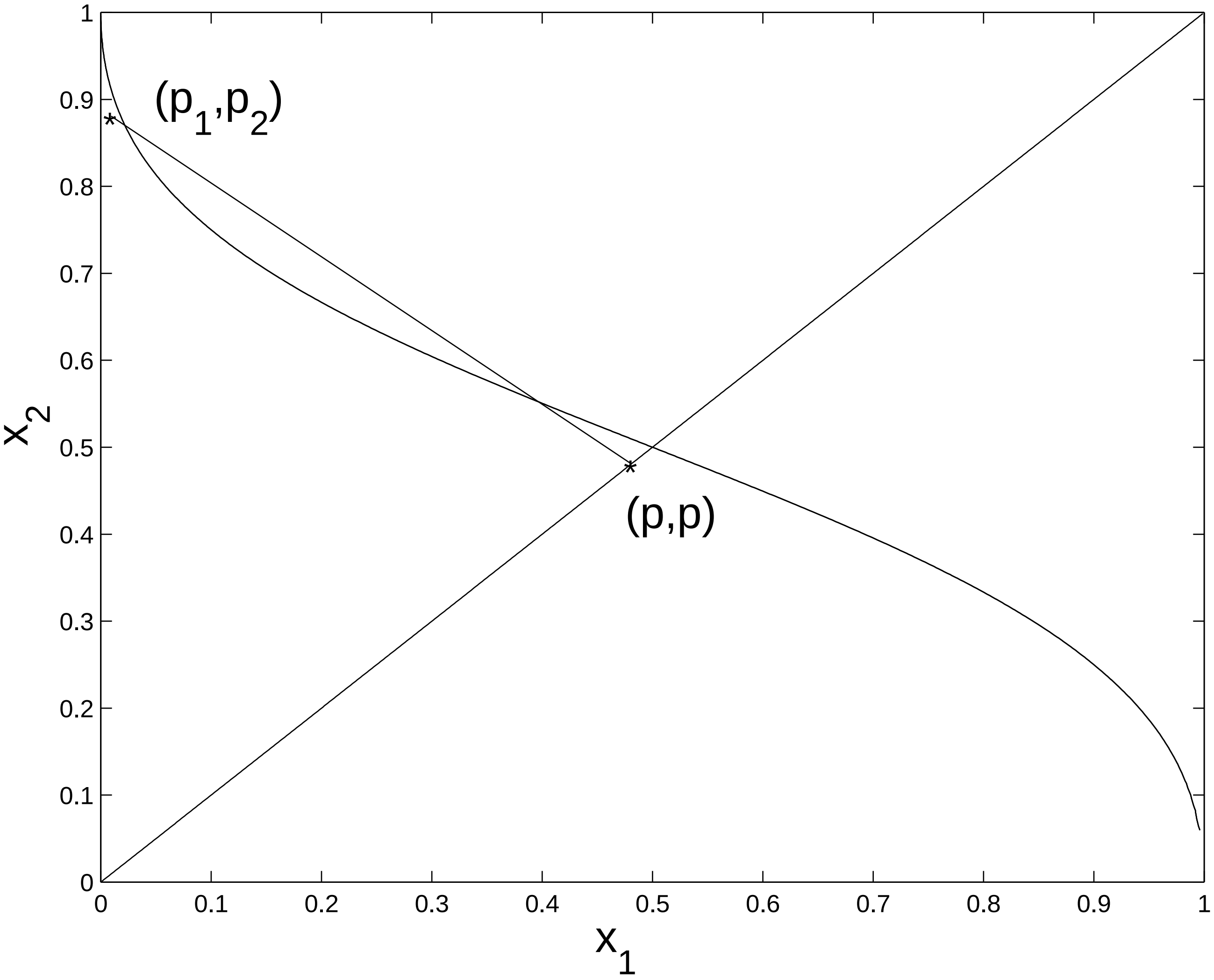
$$\left(\frac{(1-q_1)(1-q_2)^2}{q_1 q_2^2}\right) = 217/300 < 1,$$

which shows that, with these parameters, Games A and B are losing, but the randomised game in which Games A and B are both played with probability $1/2$ is winning.

The Ubiquity of Parrondo’s Paradox

Observe that game A is a version of game B with $p_1 = p_2 = p$. Furthermore, game C is another parametrisation of game B with parameters $q_1 = \gamma p + (1 - \gamma)p_1$ and $q_2 = \gamma p + (1 - \gamma)p_2$. Thus, all three games are versions of game B, but with different values of the parameters.

Because of this, we can study the situation by examining the regions in parameter space that game B is winning, losing or fair.


(p_1,p_2)
(p,p)
x_2
x_1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

The Ubiquity of Parrondo’s Paradox

Consider any game in which there is a concept of “current state” and where the movement of game depends stochastically on the values of a number of real parameters $\{x_1, \ldots, x_n\}$.

Let $\mathcal{X}$ denote the subset of R^n consisting of those values of $\{x_1, \ldots, x_n\}$ which are consistent with the rules of the game and suppose that $\mathcal{X}$ can be partitioned by an $n-1$ dimensional manifold $\mathcal{F}$ into winning and losing regions $\mathcal{W}$ and $\mathcal{L}$ respectively.

Points on $\mathcal{F}$ correspond to parameter values where the game is fair.

The Ubiquity of Parrondo’s Paradox

Theorem

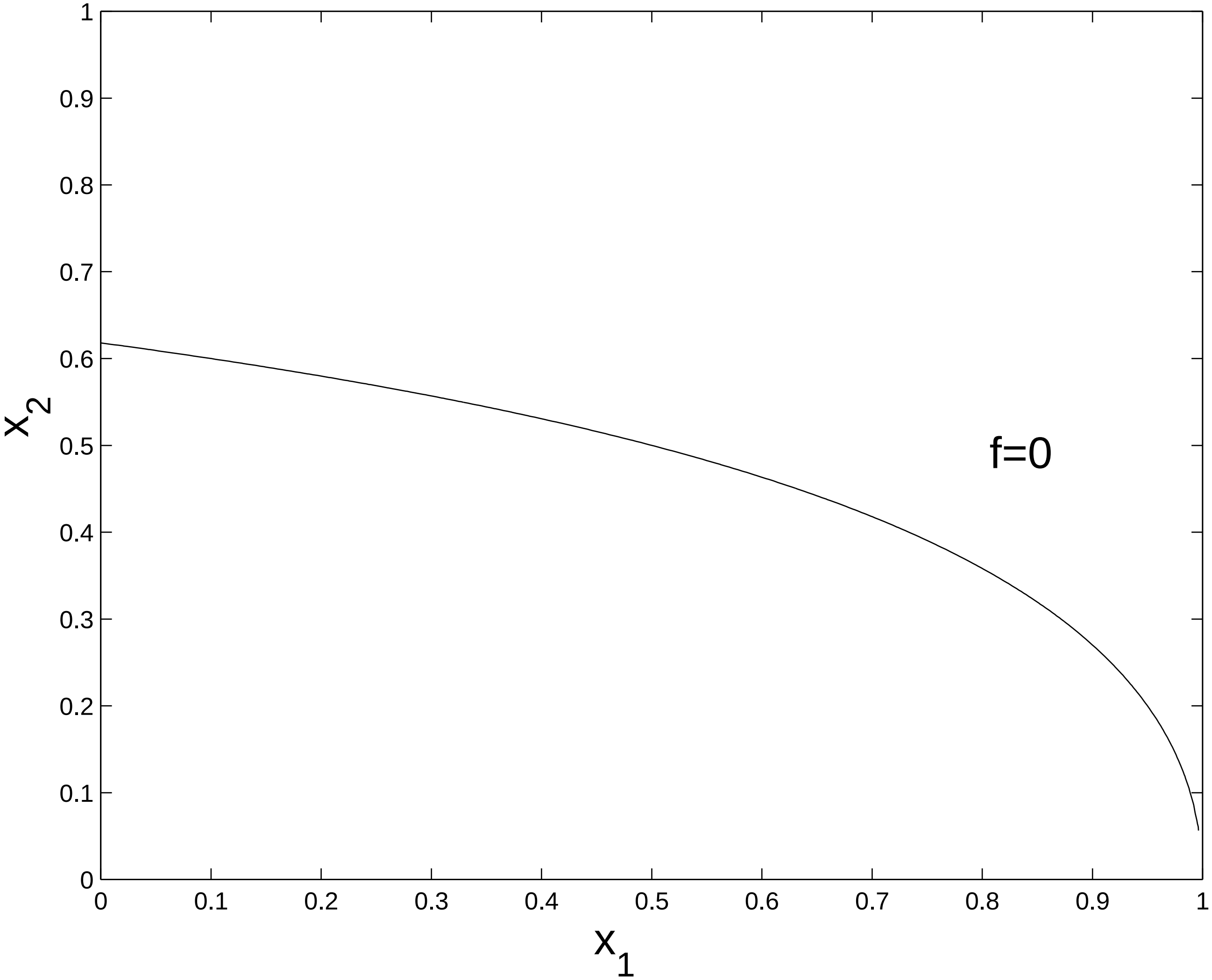
Provided that the region $\mathcal{F}$ of parameter values for which the game is fair is not a hyperplane, it is possible to demonstrate a Parrondo’s paradox.

An Example

Consider a game in which the player

- wins with probability x_2 if their previous two games have been wins,
- wins with probability x_1 otherwise.

By taking $x_2 < x_1$ in this game, we can model the type of dependency that many gamblers erroneously assume is present in sequences of independent games; that a player is less likely to win after a sequence of wins.



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x_2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x_1
f=0