Alternating search at two locations

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Asymmetric rendezvous on the line

- Two players placed a distance d apart on the line and faced in random directions
- d chosen from a known distribution, $d \sim G$
- Each player has maximum speed $\frac{1}{2}$
- Objective is to meet in minimum expected time to meet, $R\left(G\right)$

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- Each player has maximum speed $\frac{1}{2}$
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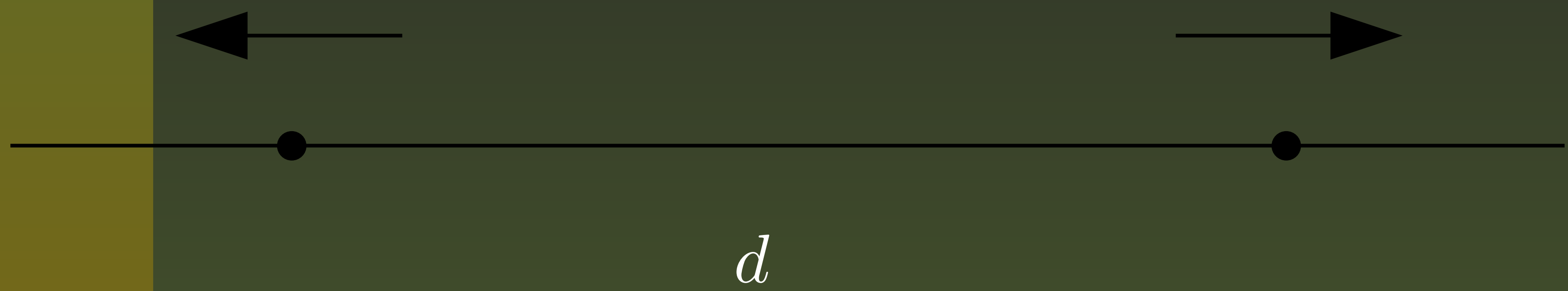
- Asymmetric
- Line not oriented

The problem



Two special cases

- (i) d is known to be 2 (a point distribution)
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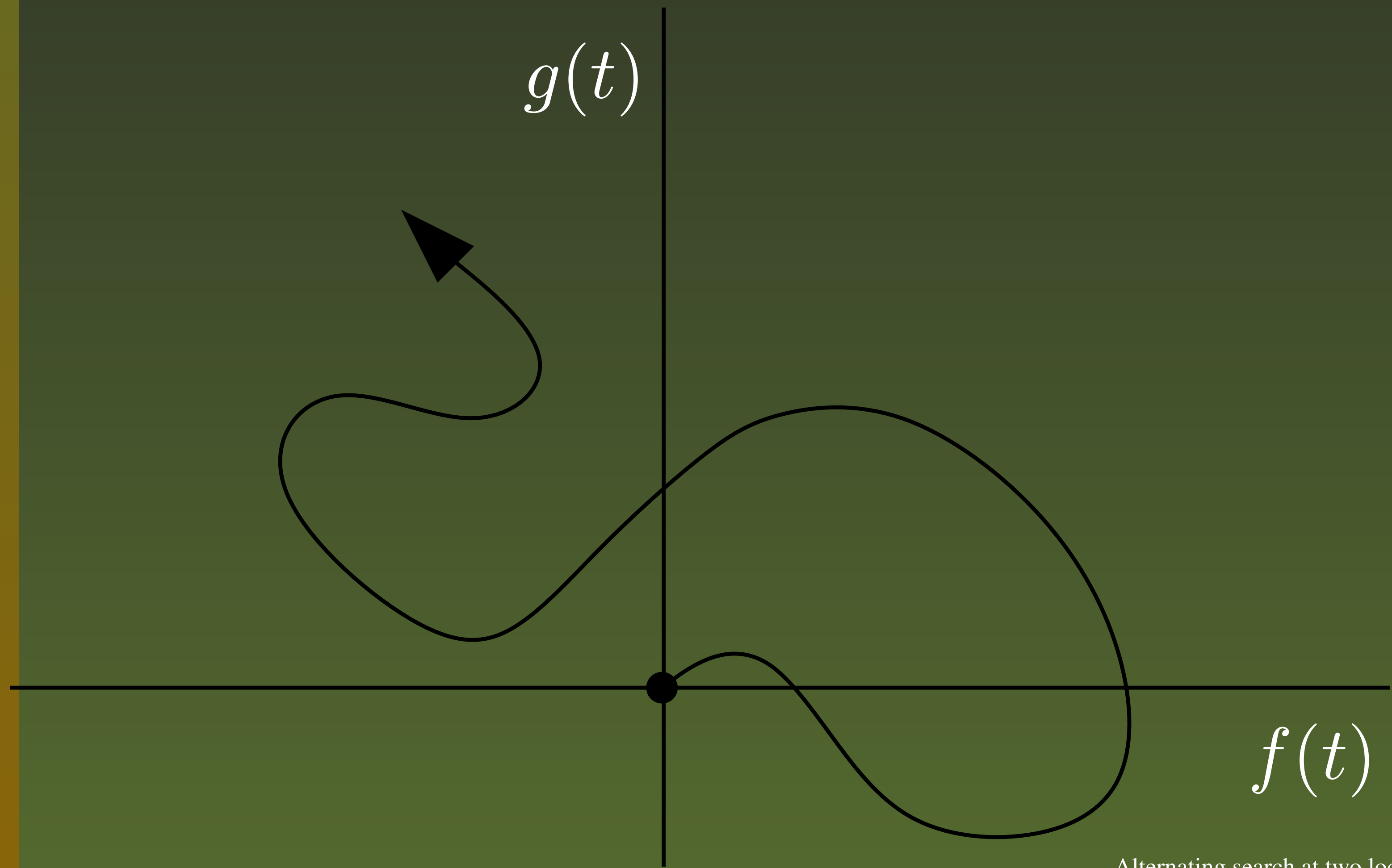
- Even (i) unsolved in symmetric case

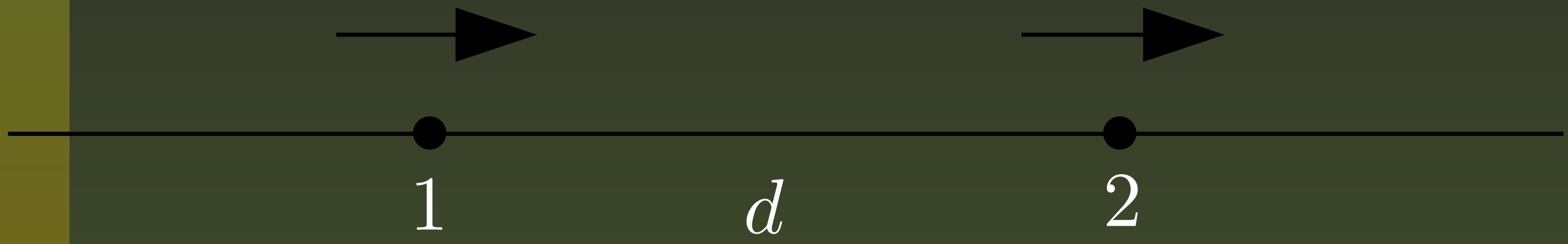
Movement in state space

If $P1$ follows trajectory $f(t)$ from his starting position, and $P2$ $g(t)$, we can plot $(f(t), g(t))$.



One starting situation

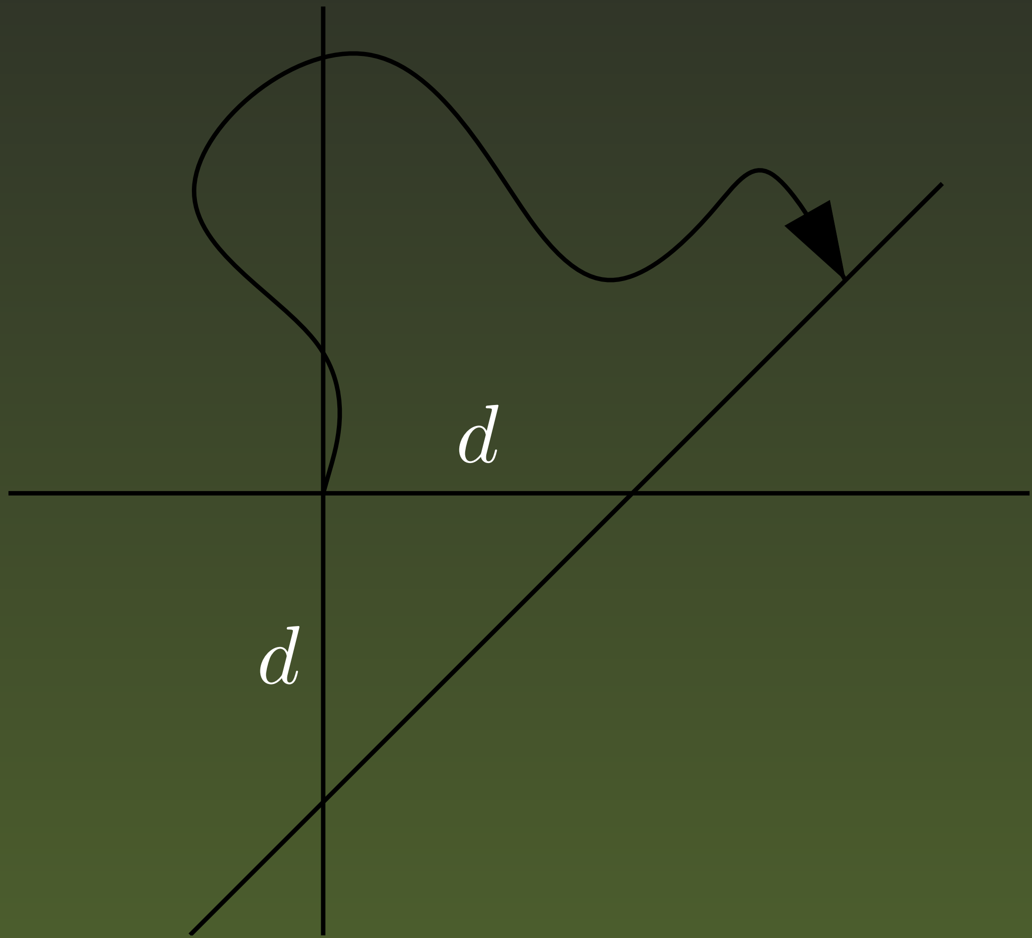
Suppose they start:



they will meet when

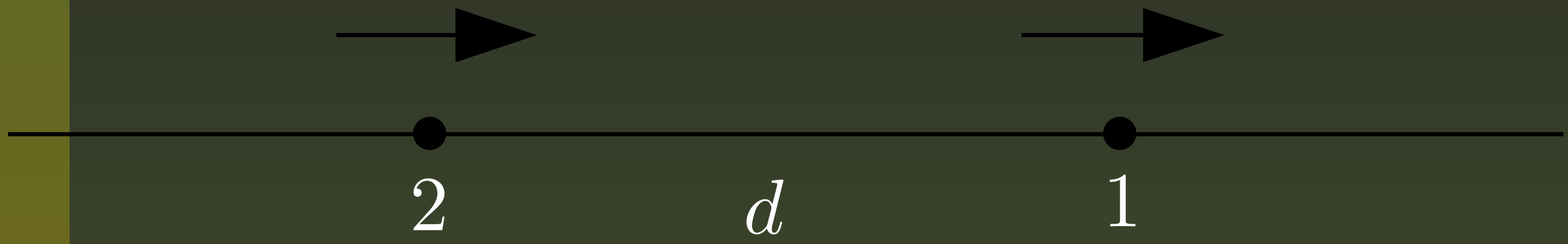
$$f(t) = g(t) + d$$

Meeting



$$f(t) - g(t) = d$$

Other starting situations



$$g(t) - f(t) = d$$

Similarly:

$$f(t) + g(t) = d$$
$$-f(t) - g(t) = d$$

Objective function

Define:

$$z_1 = f(t) + g(t); \qquad z_2 = -f(t) + g(t)$$

Then:

$$P_{f,g}(s) = \frac{1}{4} G\left(\max_{0 \le t \le s} z_1(t)\right) + \frac{1}{4} G\left(\max_{0 \le t \le s} -z_1(t)\right) + \frac{1}{4} G\left(\max_{0 \le t \le s} z_2(t)\right) + \frac{1}{4} G\left(\max_{0 \le t \le s} -z_2(t)\right)$$

Objective:

$$R(G) = \min_{f,g} \int_0^\infty s \, dP_{f,g}$$

Double linear search problem

- Alpern and Beck, 1997
- Search for a stationary object hidden on one of two lines
- Object hidden at d from the origin
  - $d \sim G$
  - equally likely in either direction
  - equally likely on either line
- Control 2 searchers, one on each line, starting at origin
- Combined speed of searchers cannot exceed 1

The new problem

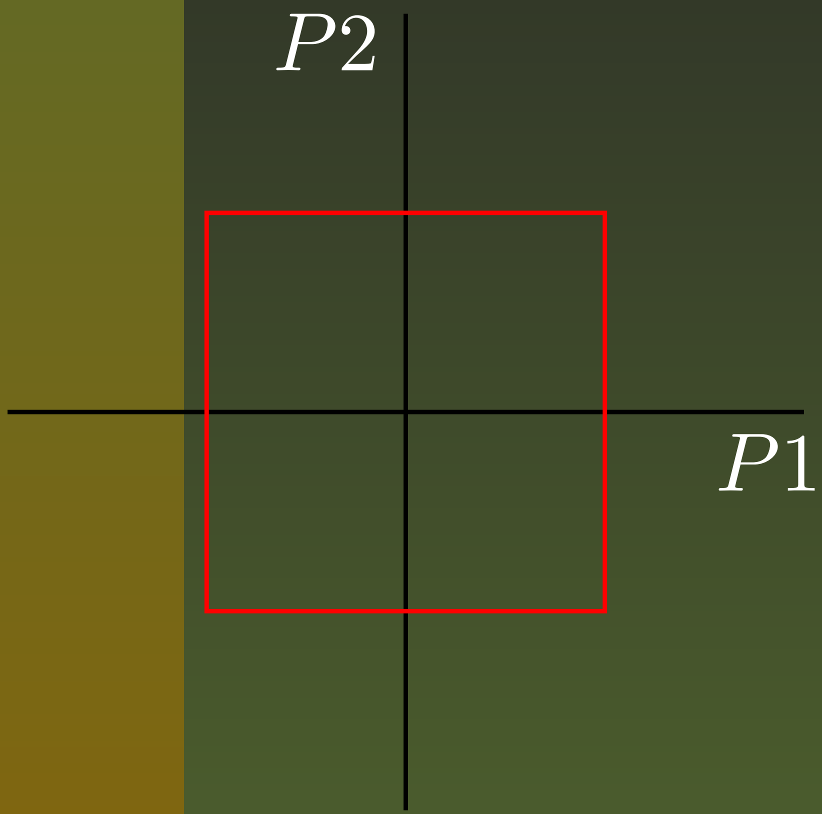
Probability of success by time s

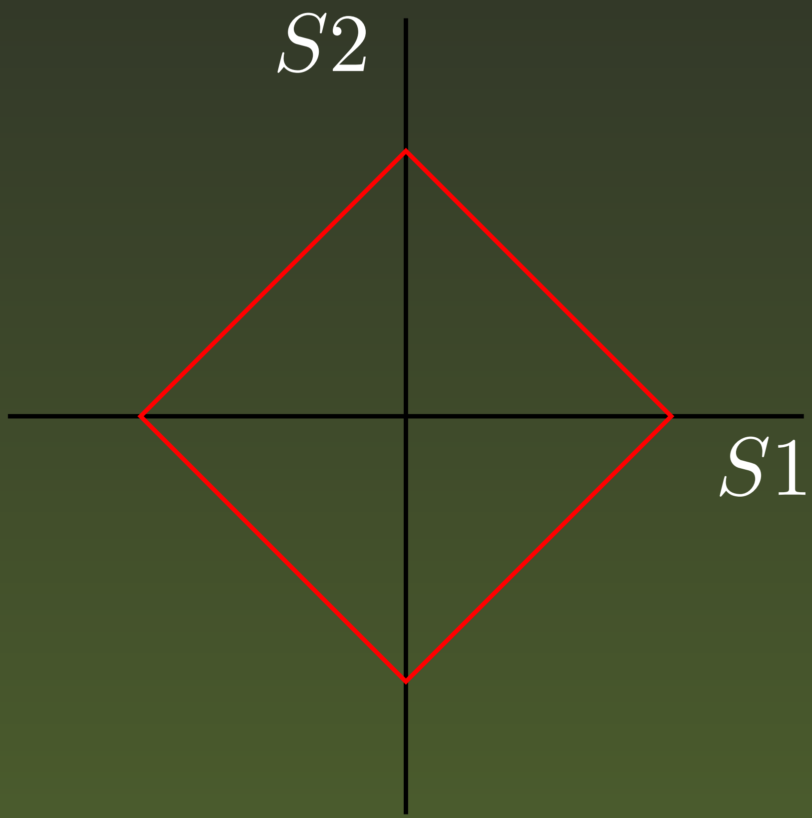
Probability that object has been found by s is

$$P_{z_1,z_2}(s) = \frac{1}{4}G\left(\max_{0\le t\le s} z_1(t)\right) + \frac{1}{4}G\left(\max_{0\le t\le s} -z_1(t)\right) + \frac{1}{4}G\left(\max_{0\le t\le s} z_2(t)\right) + \frac{1}{4}G\left(\max_{0\le t\le s} -z_2(t)\right)$$

The two problems are the same.

Relationship of the two problems





DLSP with $d = 2$

- Only one sensible way of searching each line
- Two ways of interleaving the two searches, but one clearly better

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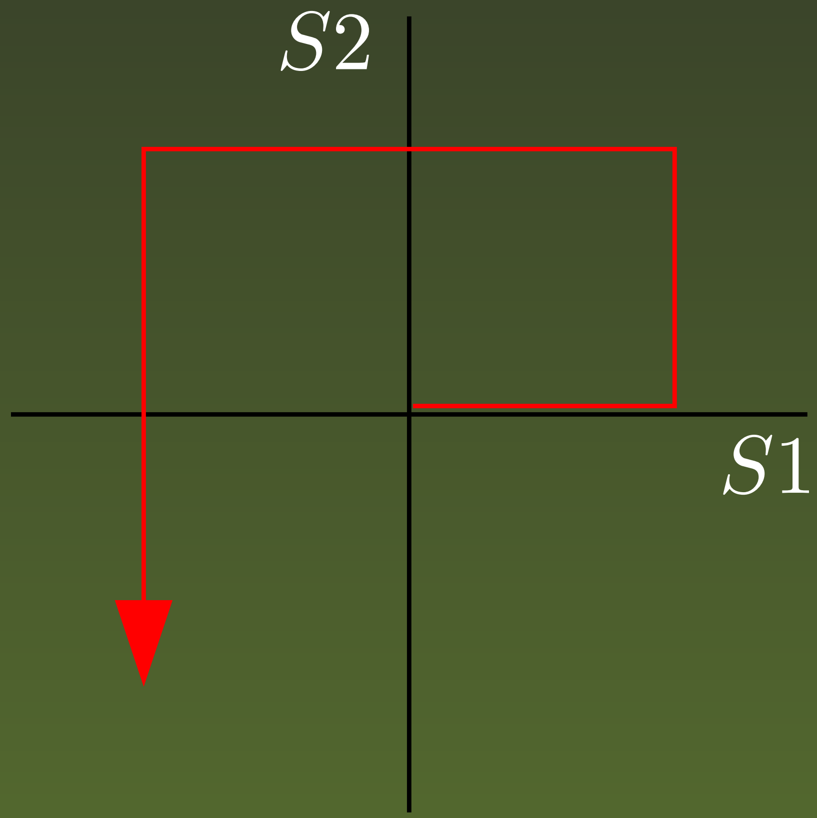

ARSPL with $d = 2$

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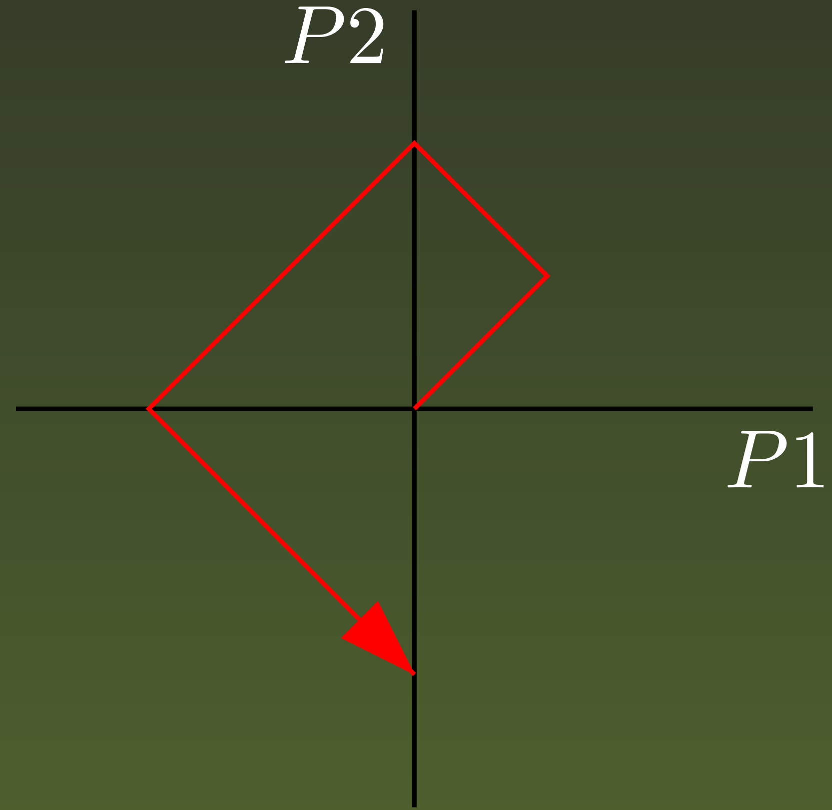
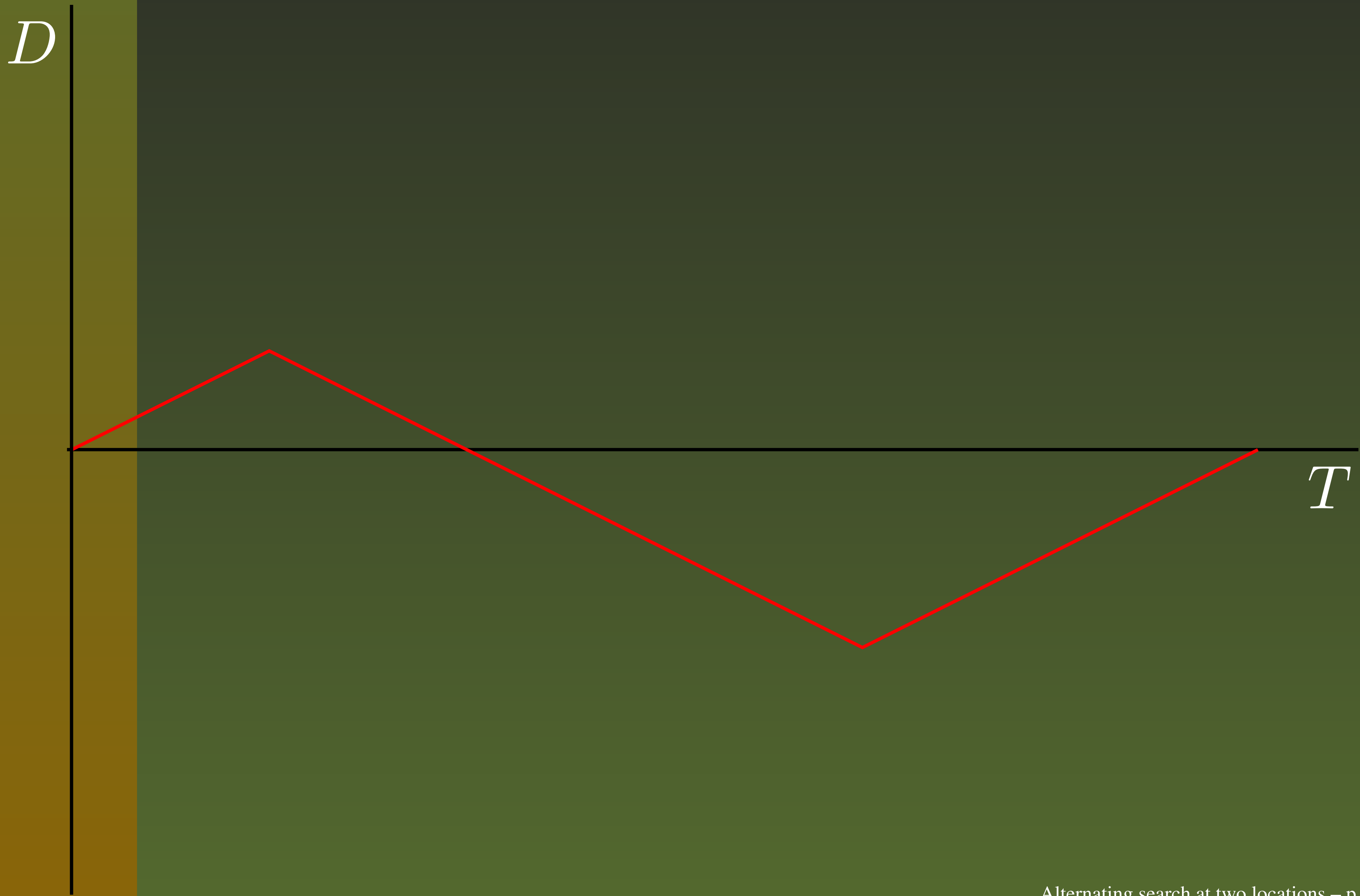
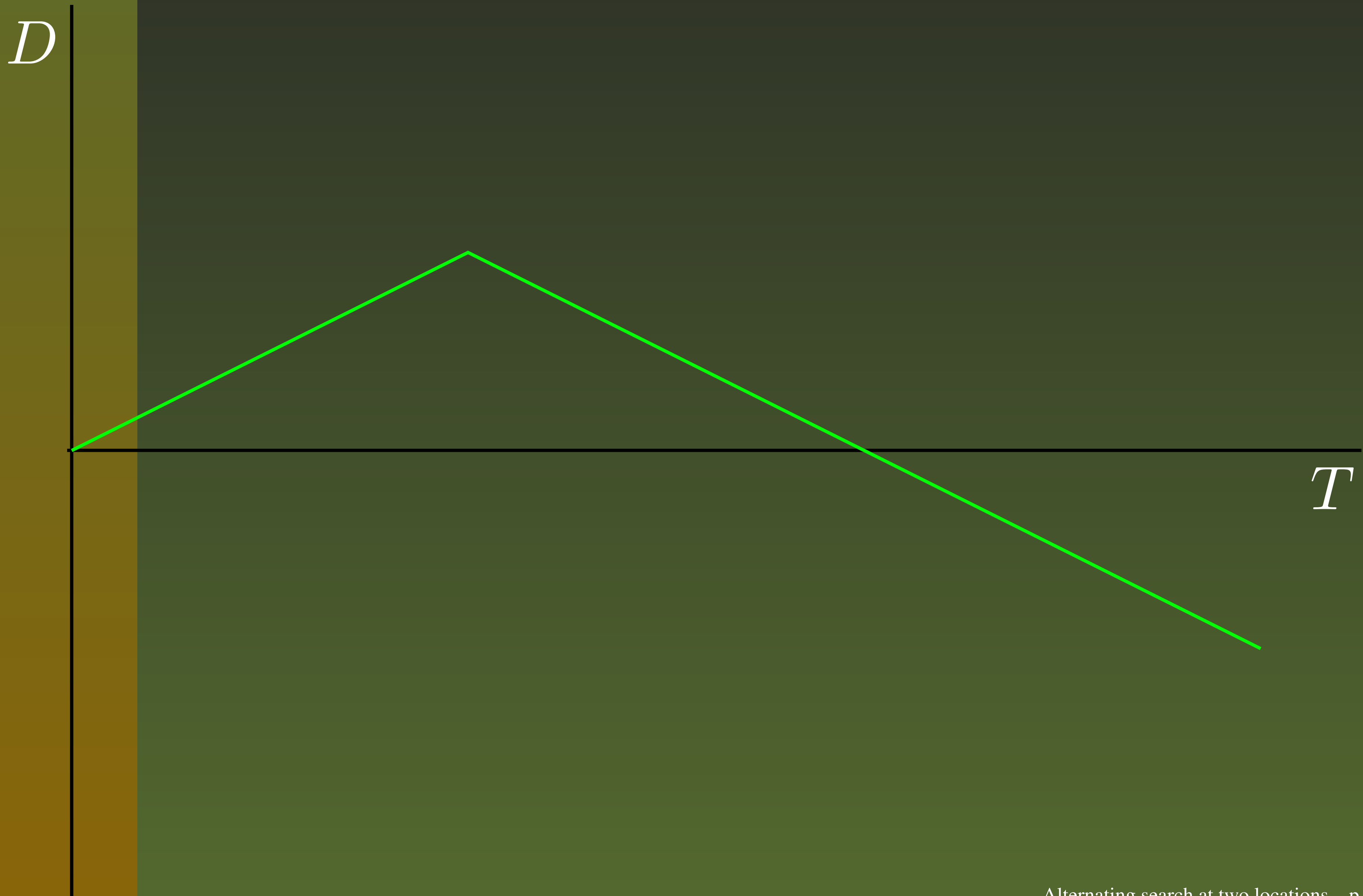


ARSPL with $d = 2$: player 1 strategy

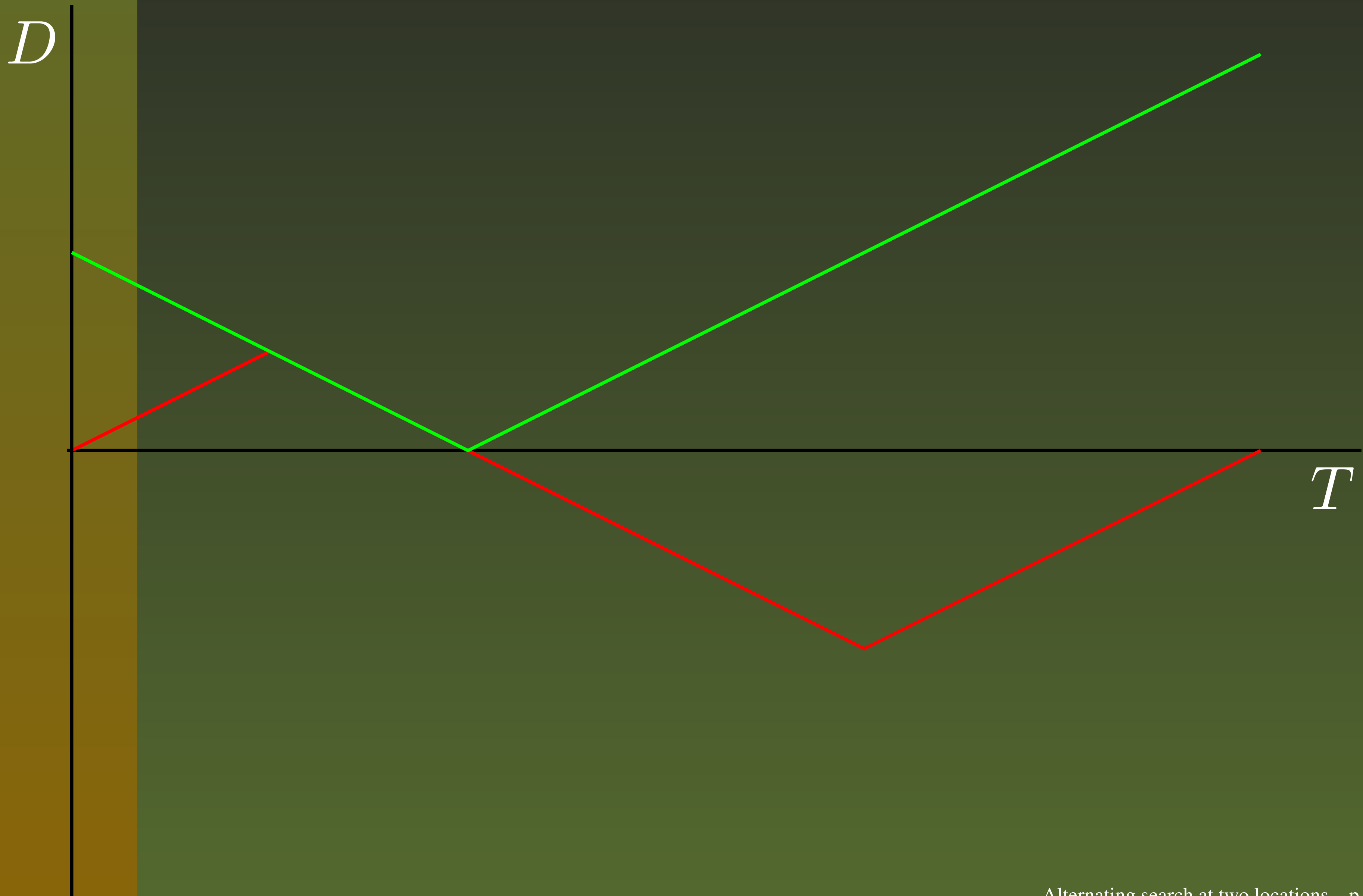


ARSPL with $d = 2$: player 2 strategy

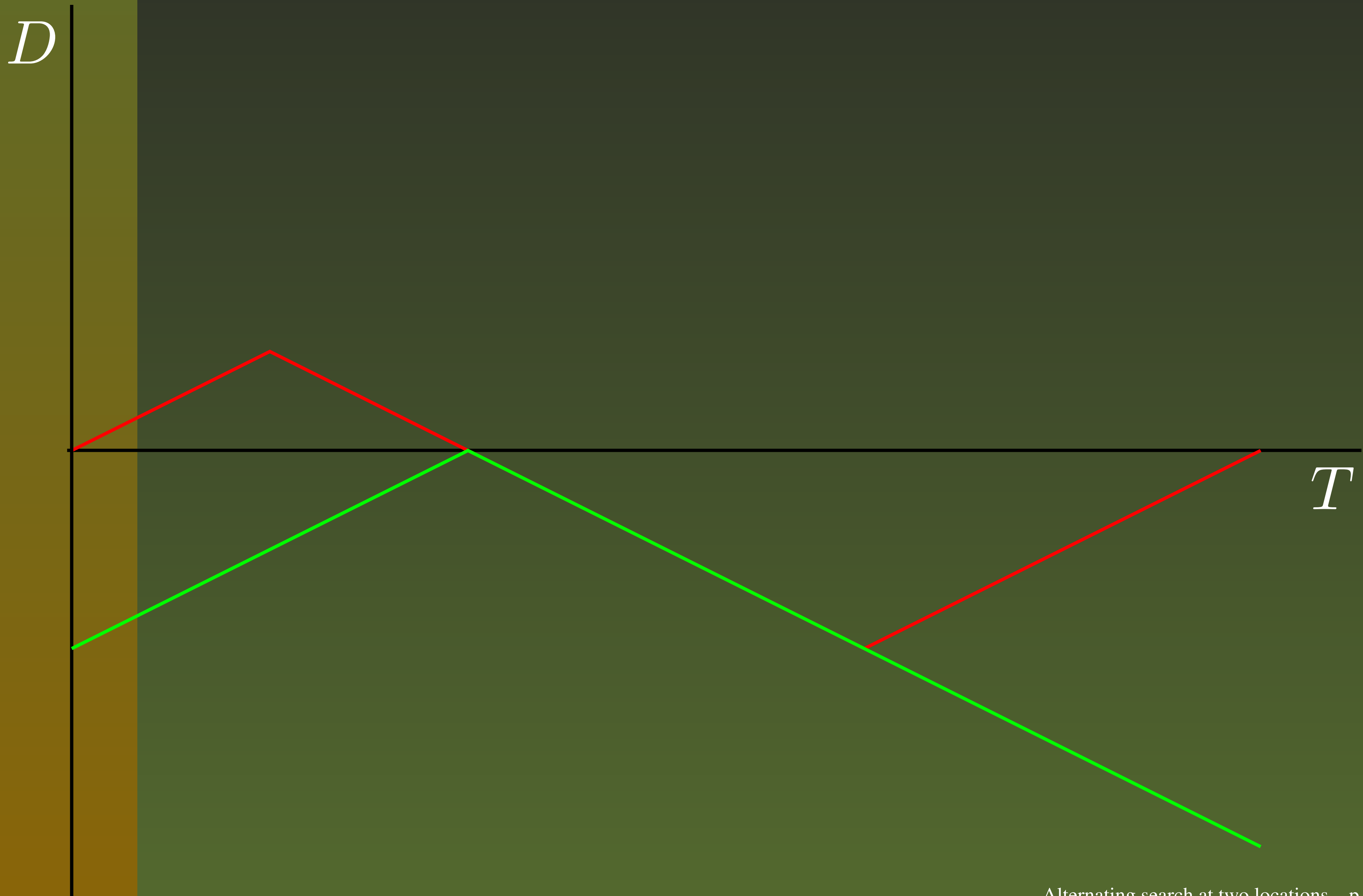


ARSPL with $d = 2$: start 1

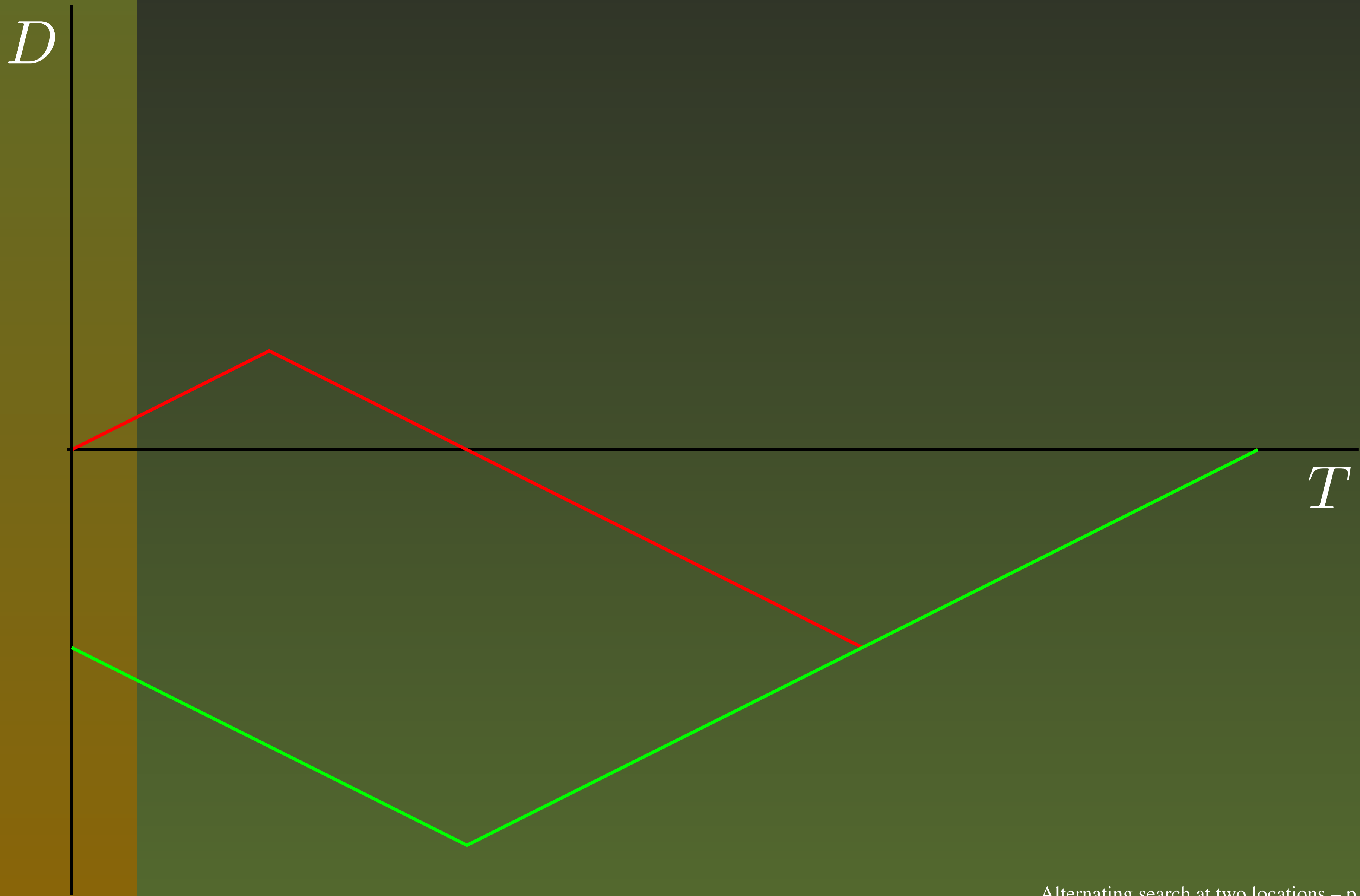
ARSPL with $d = 2$: start 2



ARSPL with $d = 2$: start 3



ARSPL with $d = 2$: start 4



Searching two locations

1. Decide how to search each region (line, circle, graph, etc.) separately
2. Calculate how to alternate between the two regions so as to minimise the expected time to find the object

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1. Decide how to search each region (line, circle, graph, etc.) separately
2. Calculate how to alternate between the two regions so as to minimise the expected time to find the object
3. Once 1 is decided, we are effectively searching two rays
4. If we search the first ray for time t, the probability we will find it is $F_1(t)$
5. We are interleaving two probability distributions

Searching two rays in discrete time

- Two locations, each with 5 boxes
- In each location boxes must be searched in fixed order

Location 1	0	1	0	0	1
Location 2	0	0	0	1	1

- Searching a box takes 1 time unit
- Boxes marked 0 have no chance of containing the object
- Boxes marked 1 have equal chance of containing the object
- Can switch location at any time for no cost

Solution

Objective: minimise expected time to find object

Search 1 first:	0	1	0	0	1	0	0	0	1	1	$\frac{26}{4}$
Search 2 first:	0	0	0	1	1	0	1	0	0	1	$\frac{26}{4}$

Solution

Objective: minimise expected time to find object

Search 1 first:	0	1	0	0	1	0	0	0	1	1	$\frac{26}{4}$
Search 2 first:	0	0	0	1	1	0	1	0	0	1	$\frac{26}{4}$
Optimum:	0	1	0	0	0	1	1	0	0	1	$\frac{25}{4}$

Algorithm

Look at the average density of initial segments of the first sequence:

Sequence 1	Cumulate	Density	
0	0	0	
1	1	$\frac{1}{2}$	*
0	1	$\frac{1}{3}$	
0	1	$\frac{1}{4}$	
1	2	$\frac{2}{5}$	

Algorithm (continued)

And the second sequence:

Sequence 2	Cumulate	Density
0	0	0
0	0	0
0	0	0
1	1	$\frac{1}{4}$
1	2	$\frac{2}{5}$

Take the initial segment of maximum density: 01.
Then left with 001 and 00011.

Repeat procedure.

Algorithm (continued)

- Proof that rule works by interchange argument
- If we have a choice between blocks:

01	0011	01	01
000111	01	0011	

  we can take the blocks in any order, but once we start a block we must finish it
- Generalise to arbitrary distributions and continuous time

Generalisation

- If search location i for time t, find object with probability $F_i(t)$
- By time t searcher has spent $\alpha(t)$ in location 1 and $\beta(t)$ in 2 with

$$\alpha(t) + \beta(t) = t$$

- Probability object found by t is:

$$H_\alpha(t) = F_1(\alpha(t)) + F_2(\beta(t))$$

Generalisation (continued)

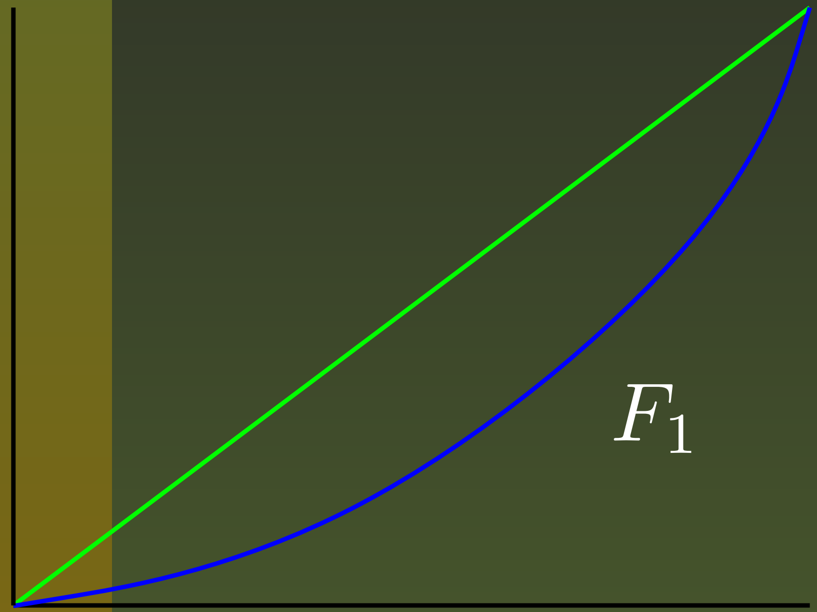
- Strategy α must be monotonic increasing and satisfy:

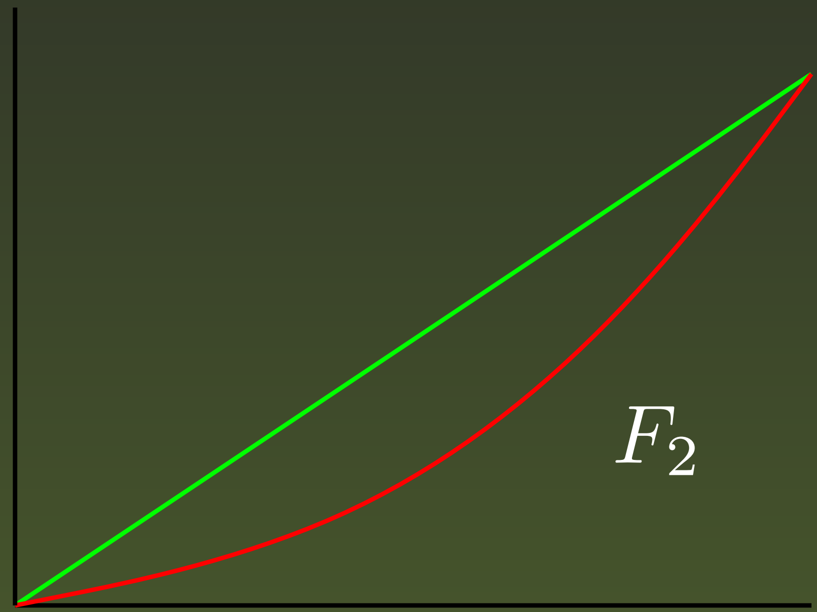
$$\alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$$
$$\alpha(0) = 0$$
$$\alpha(t) - \alpha(s) \leq t - s \qquad \forall s \leq t$$

- α is differentiable a.e. When $\alpha'(t) = 1$, the searcher is looking in location 1; when $= 0$, in 2
- Seek α which minimise $\mathrm{E}[H_\alpha]$
- Simple strategies alternate between the two locations at a sequence of times $t_1, t_2, \ldots$
- Are these all we need?

Example 1

F_1, F_2 strictly convex





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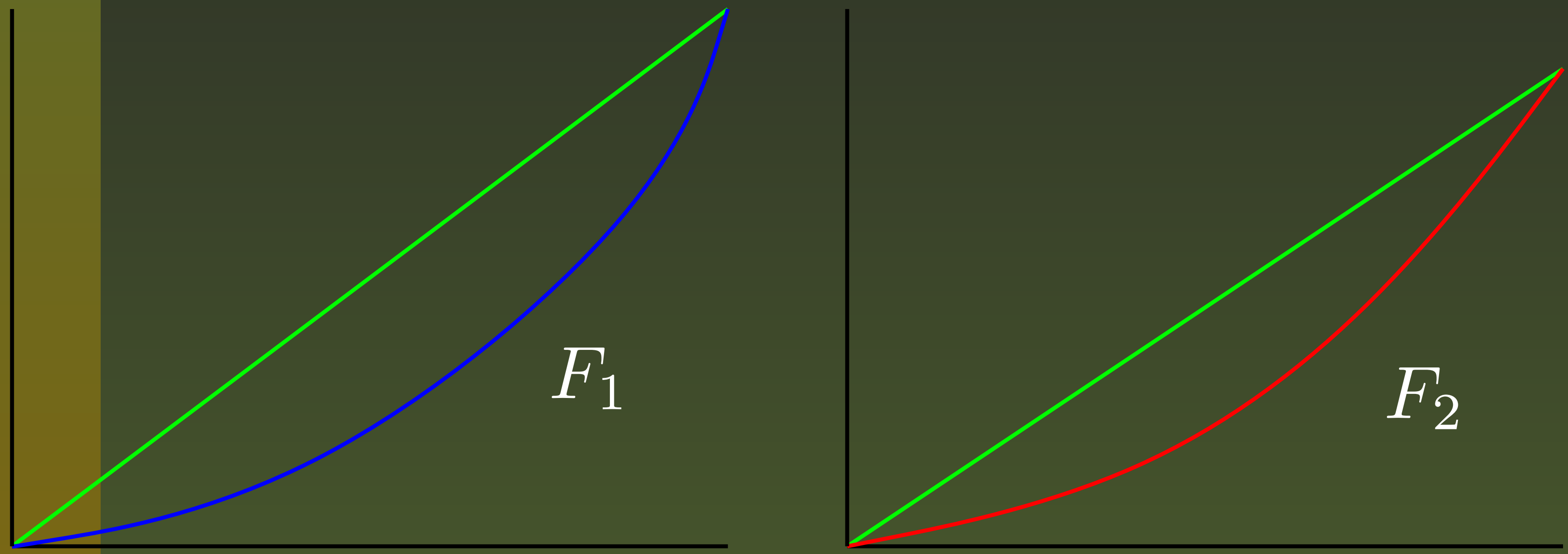
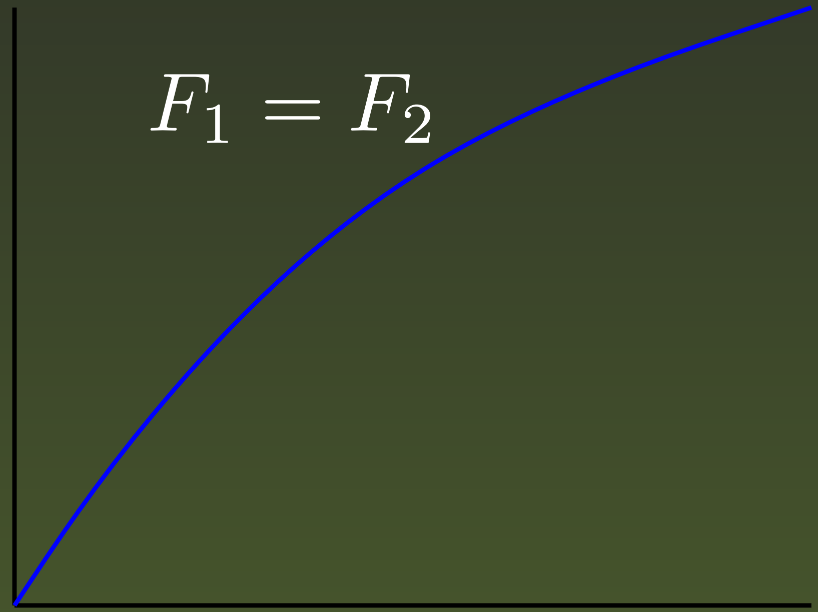


Take the location with highest density first, and search it completely. If densities equal, either can come first.

Simple strategy suffices.

Example 2

$F_1 = F_2$ strictly concave



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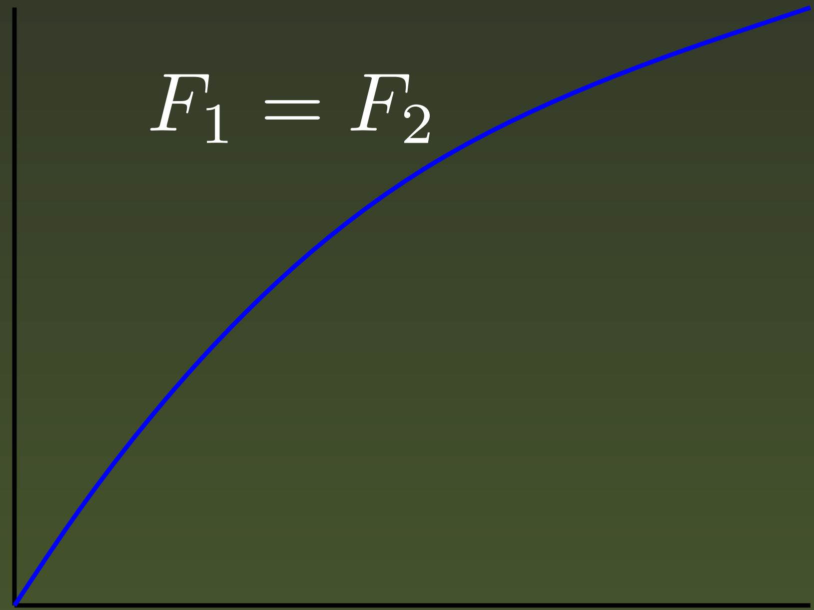
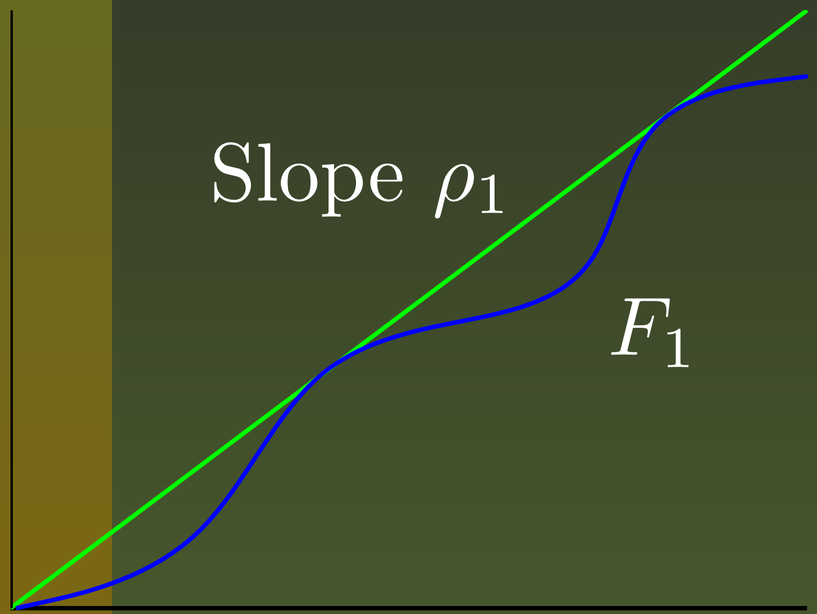


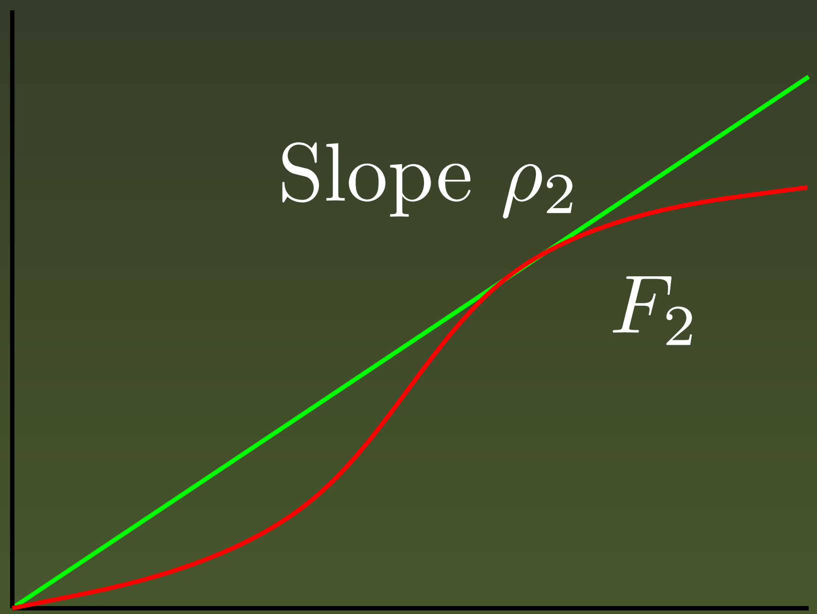
As soon as we search ϵ from location 1, location 2 has higher density. There is no simple strategy.

Optimal solution: search both simultaneously at speed $\frac{1}{2}$.

Index

In general, look at a line of slope ρ_1 which just touches F_1 from above. Similarly F_2.





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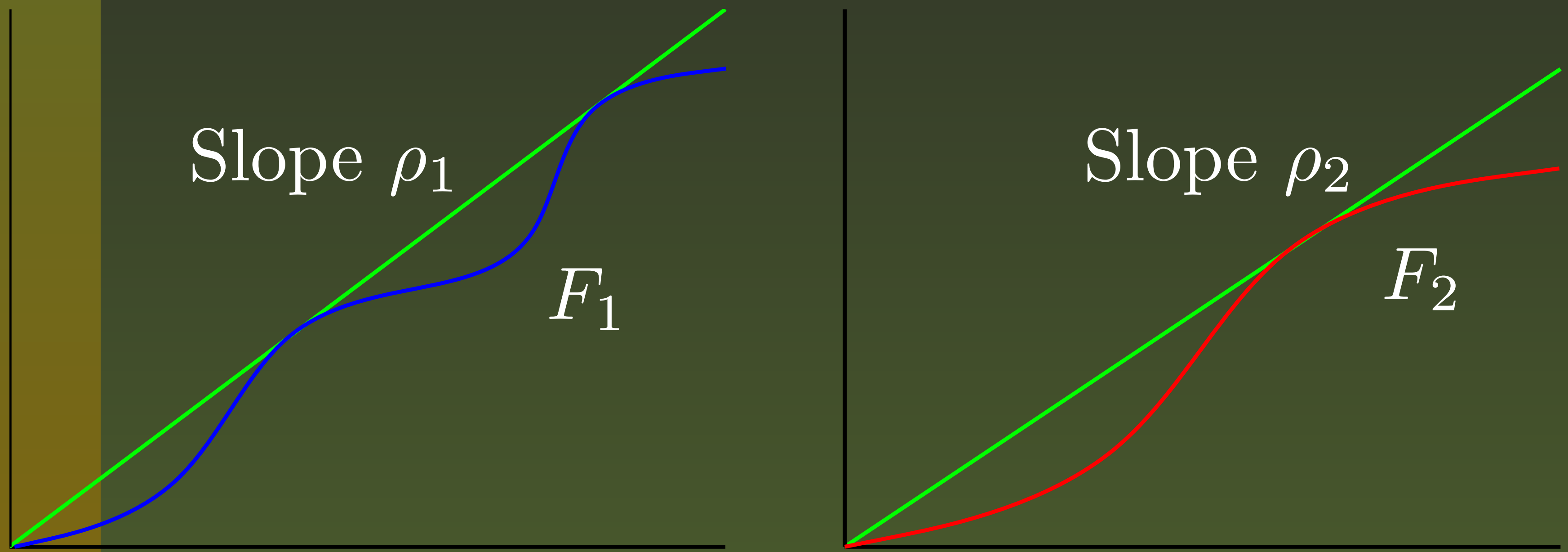


F_1 has two times with index ρ_1. F_2 has one time with index ρ_2.

Times with higher index must be searched before times with lower index.

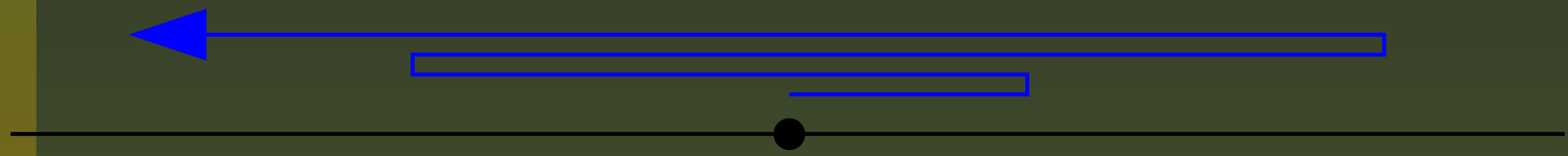
Index (continued)

- Times with equal index must be searched in blocks which touch the supporting line at each end
- Once we start a block we must finish it
- Can have infinitely blocks alternating arbitrarily close to the start:

$$\ldots A_3 B_3 A_2 B_2 A_1 B_1$$

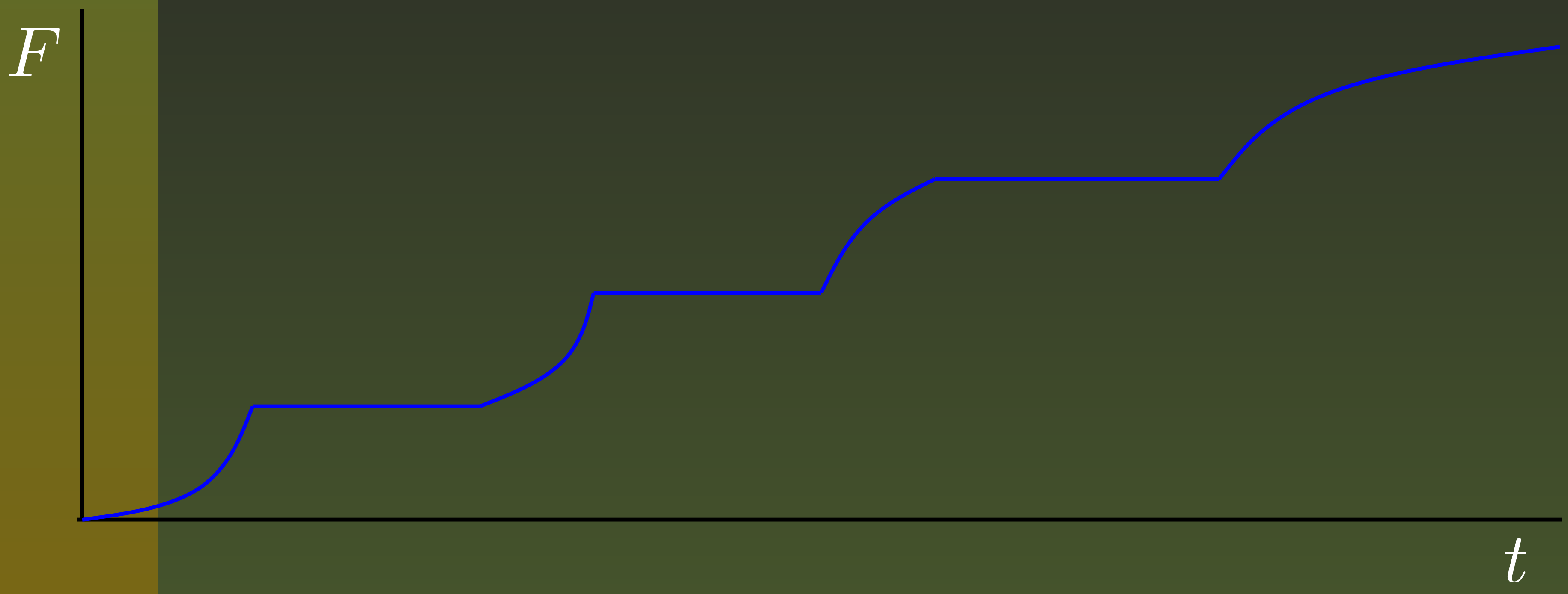
The DLSP revisited

- The only sensible strategies on one line (the LSP) are *oscillation strategies*



- We may need to oscillate infinitely often near the origin
- If we choose an oscillation strategy, we know the probability $F(t)$ of finding the object after time t

Probability of success by t



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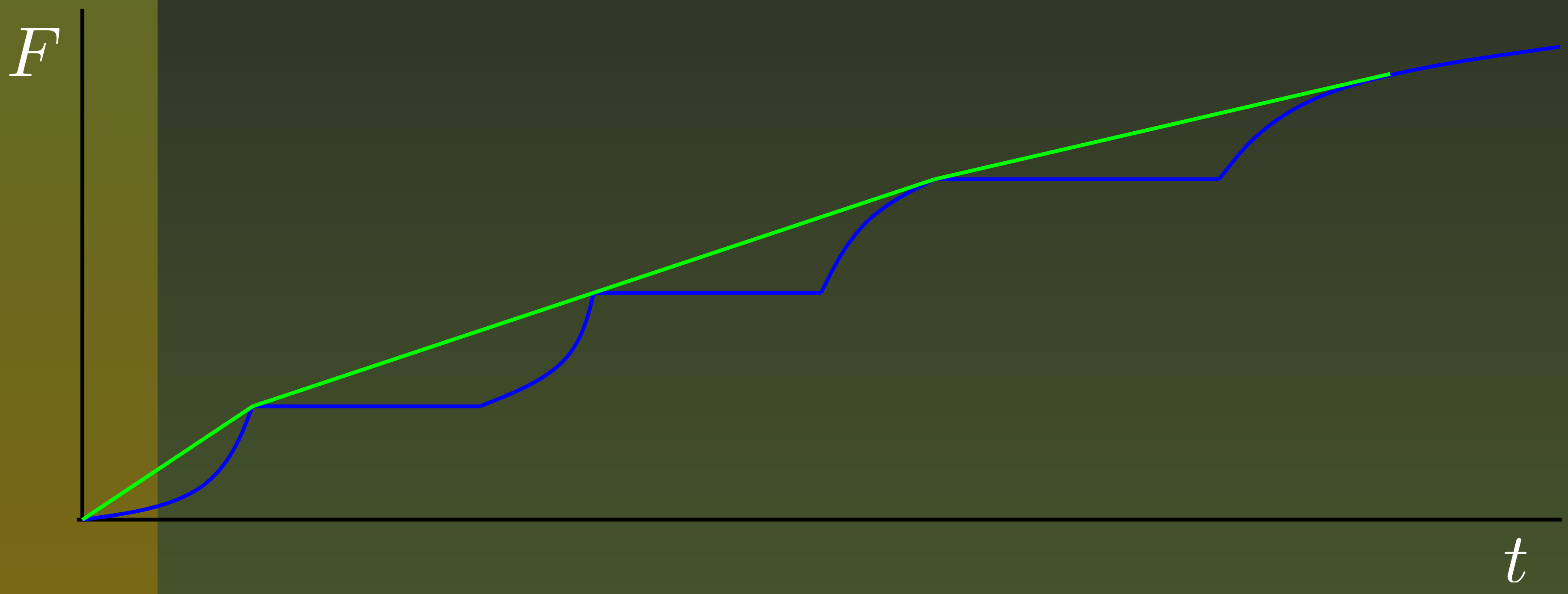

- If G is convex, a transit is never interrupted
- Having chosen the two oscillation strategies, we are back to the two ray problem

Conjecture

- For the DLSP, do we simply interleave (in an optimal way) the optimal strategies for the LSP?
- In general, if there are several ways of searching each region, do we just choose the best for each region and interleave them optimally?

The second special case

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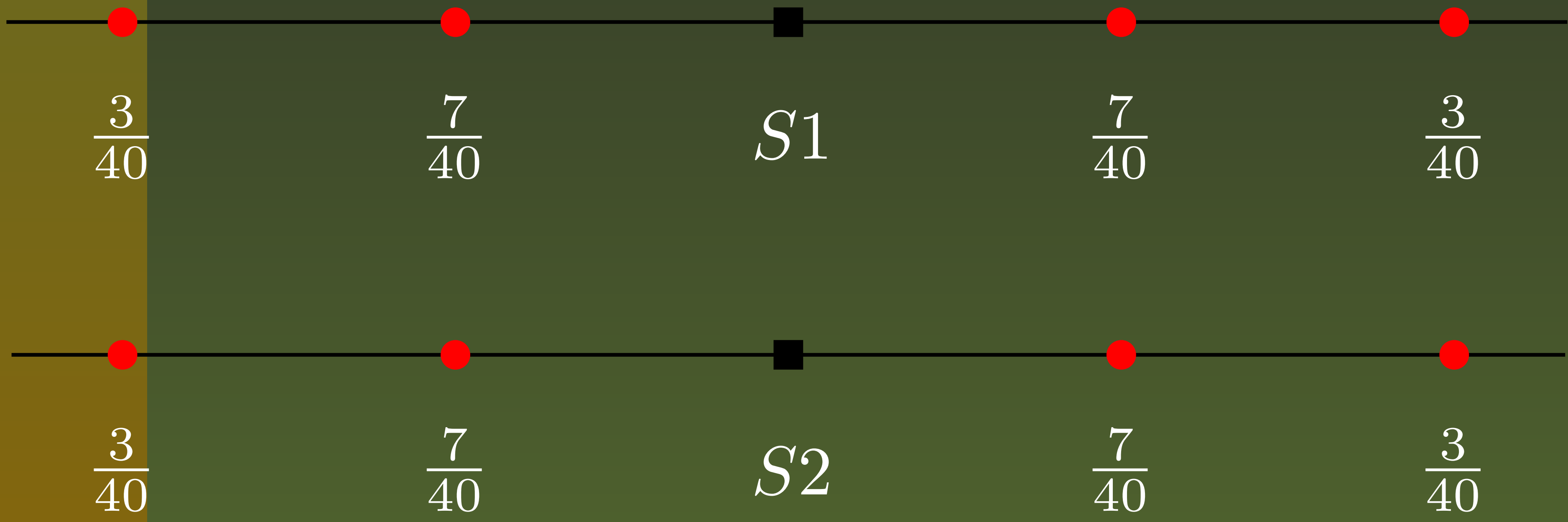

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Strategy A searches to one end of the line, then to the other.

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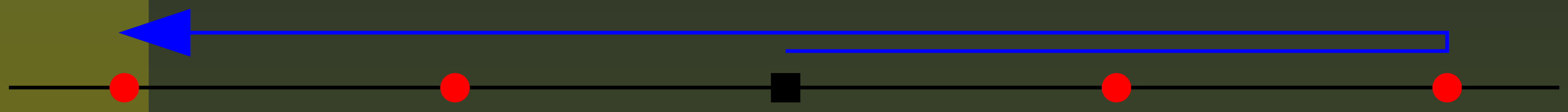

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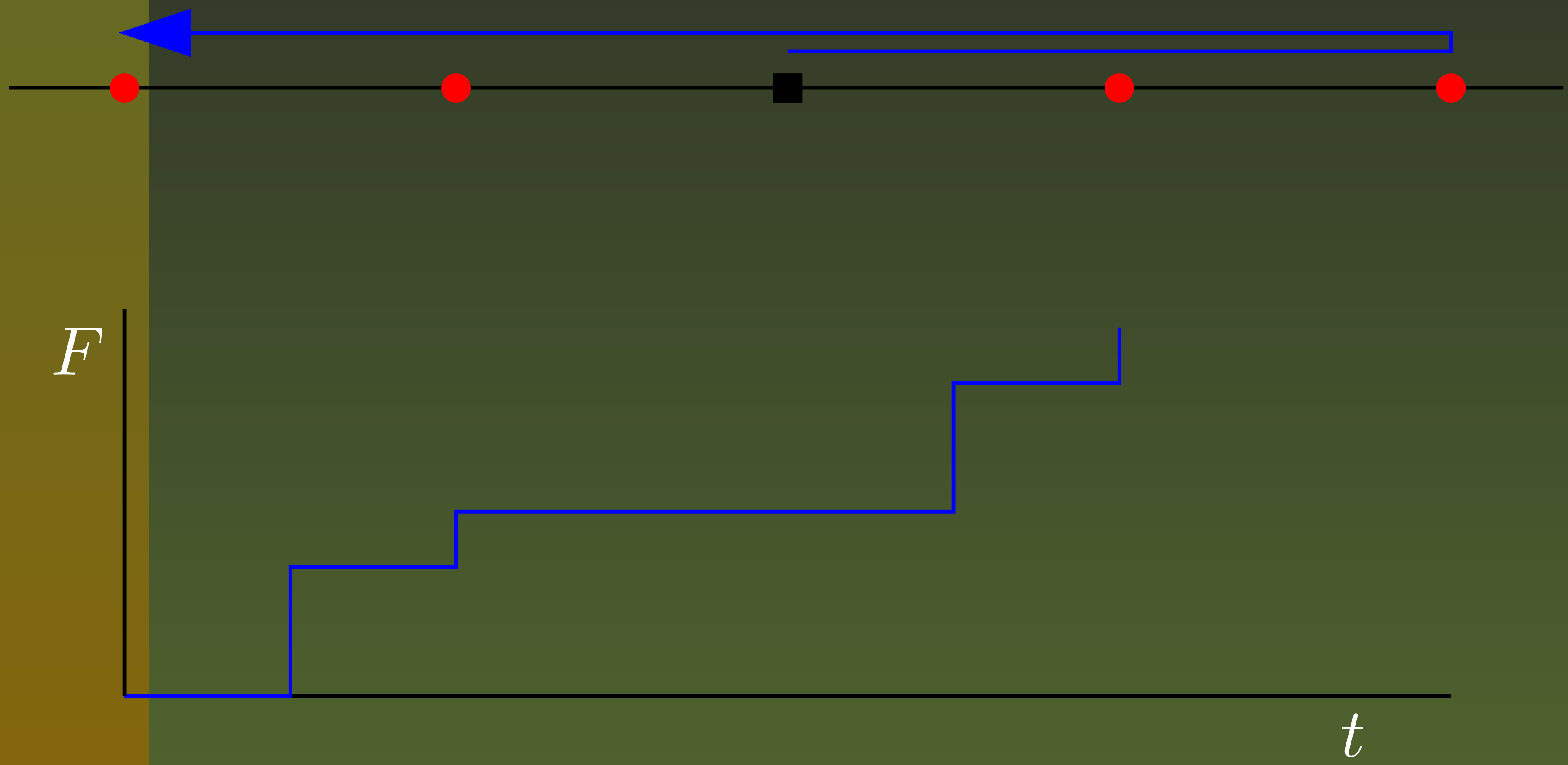

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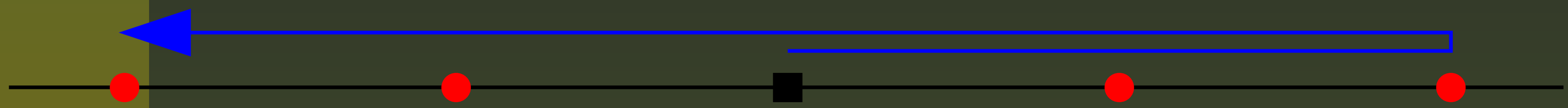

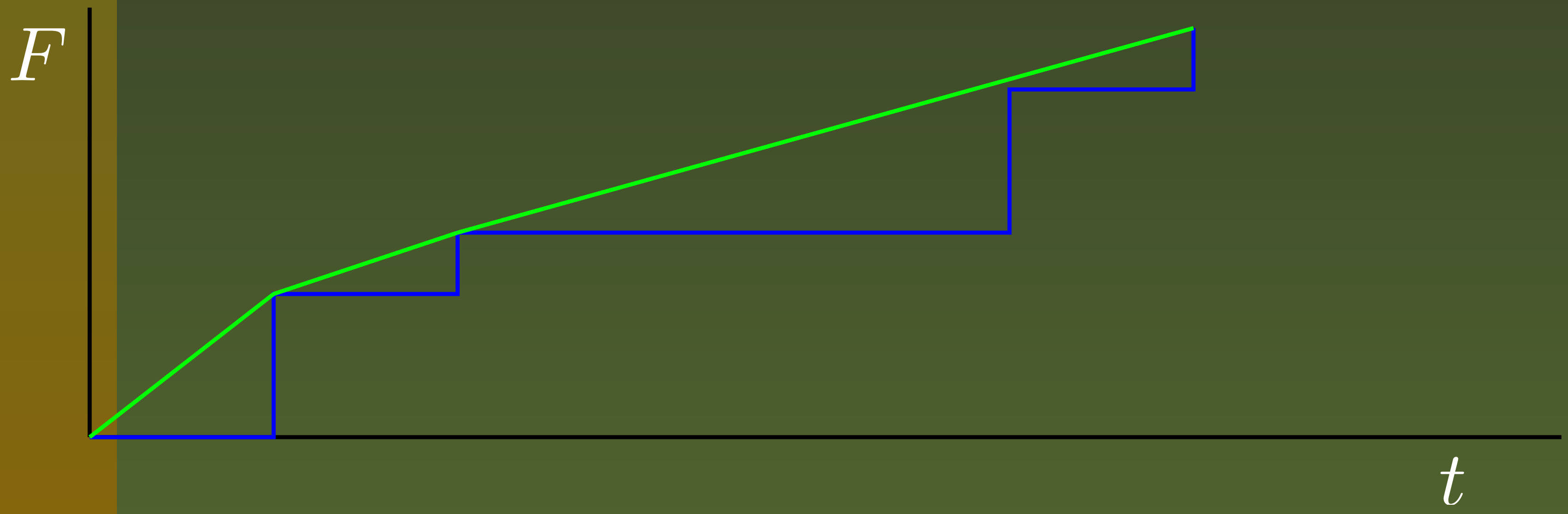

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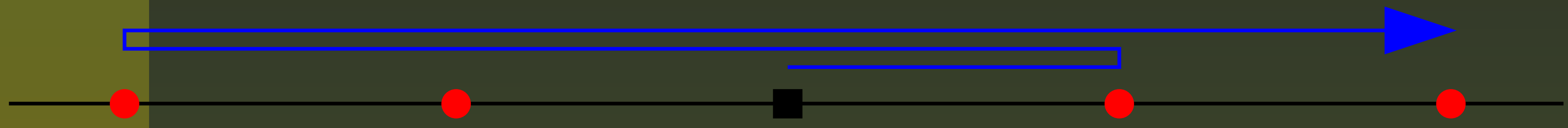

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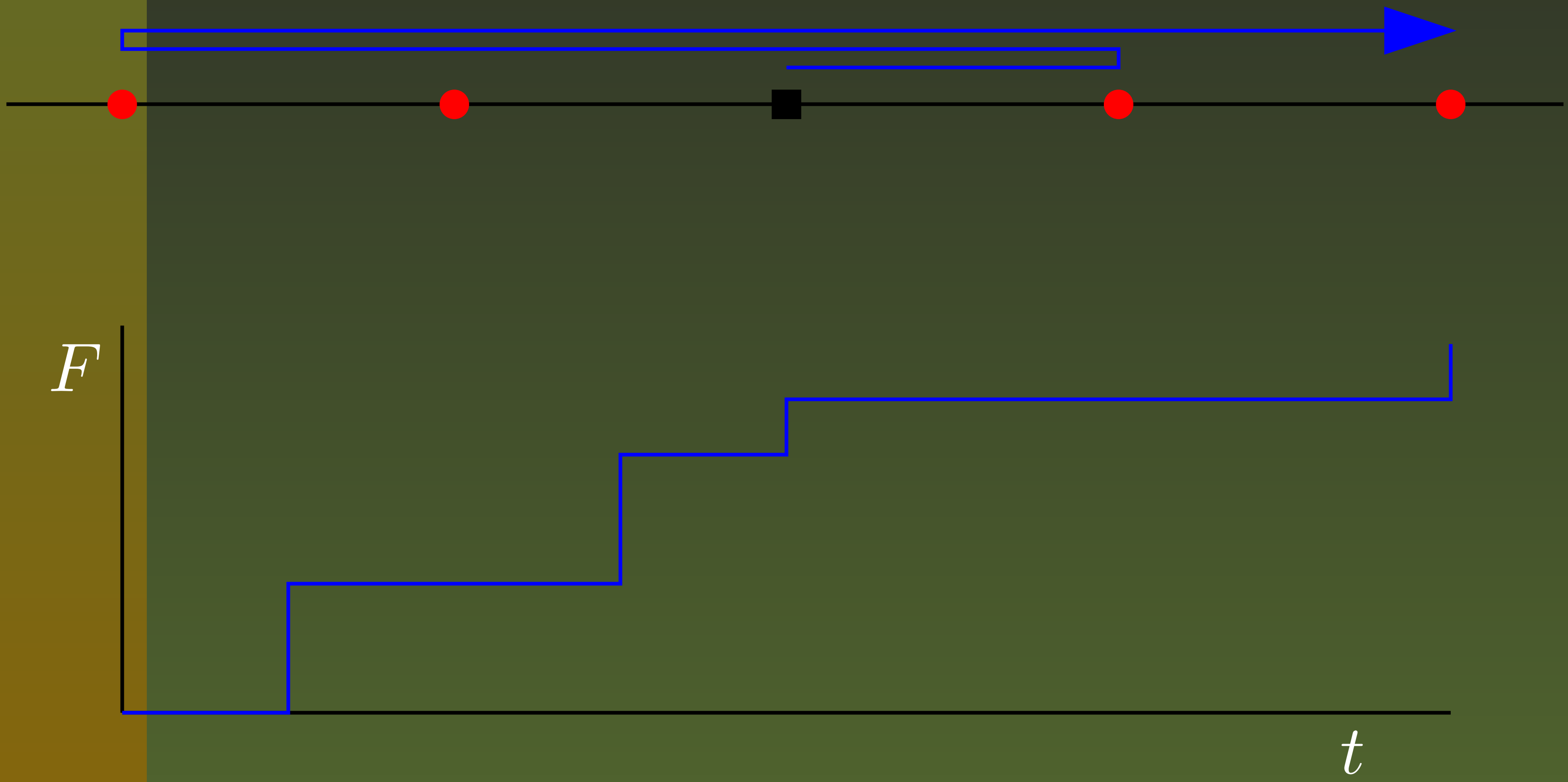

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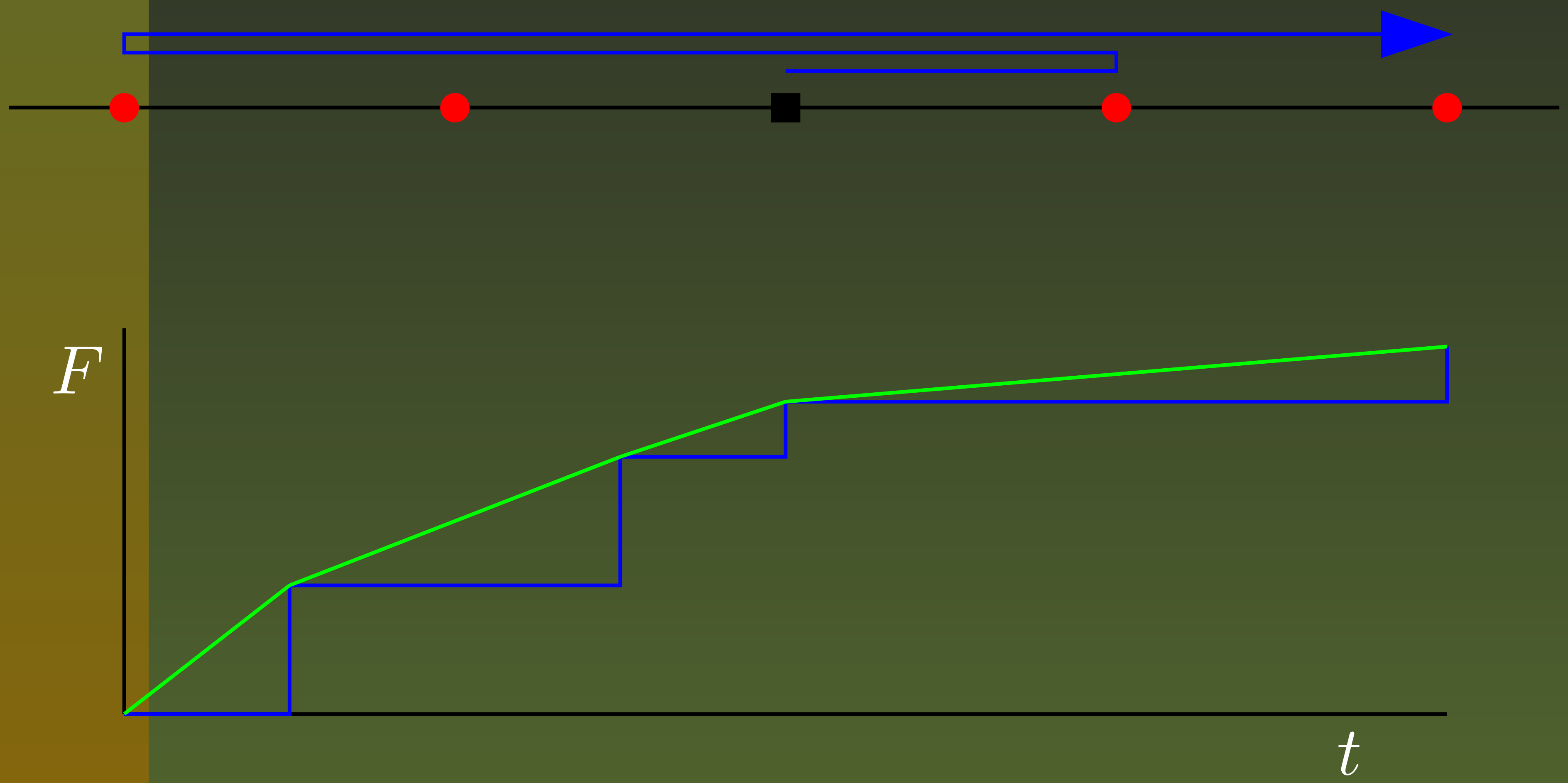

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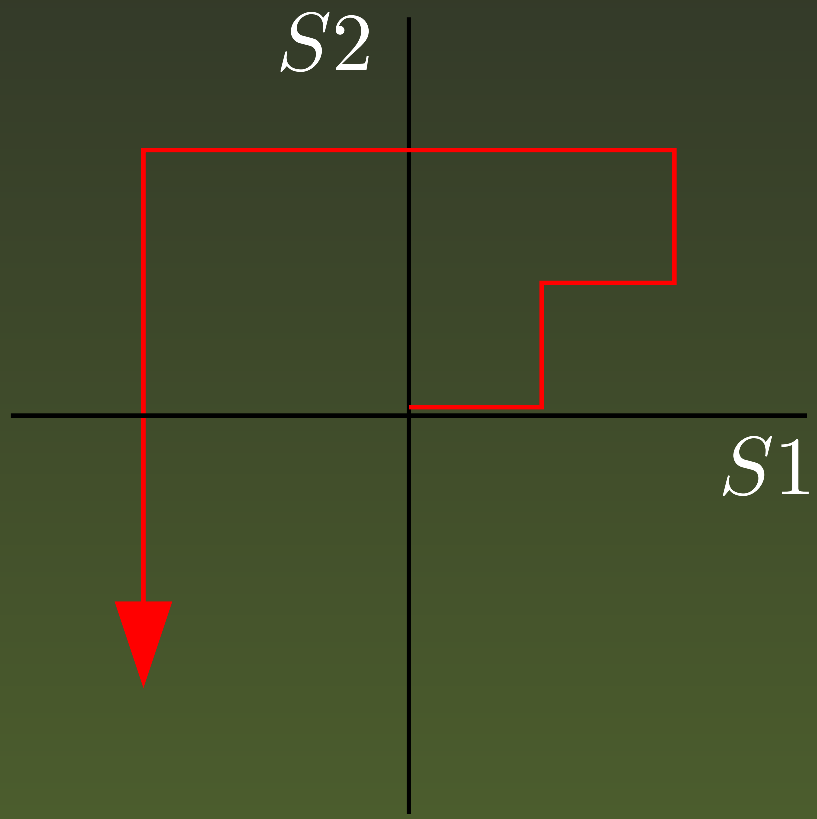

Expected time to find $= 5.7$.

Two greedy strategies

If we combine optimally we get the global strategy $[BB]_3$.

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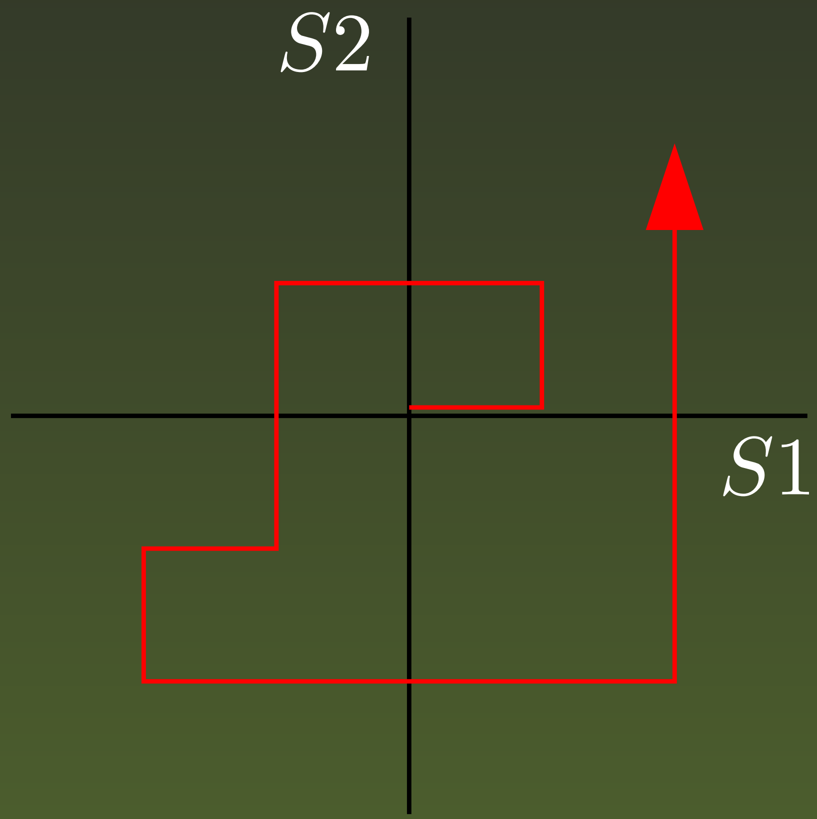

Expected time to find $= 5.5$.

One efficient and one greedy strategy

If we combine optimally we get the global strategy $[AB]_2$.

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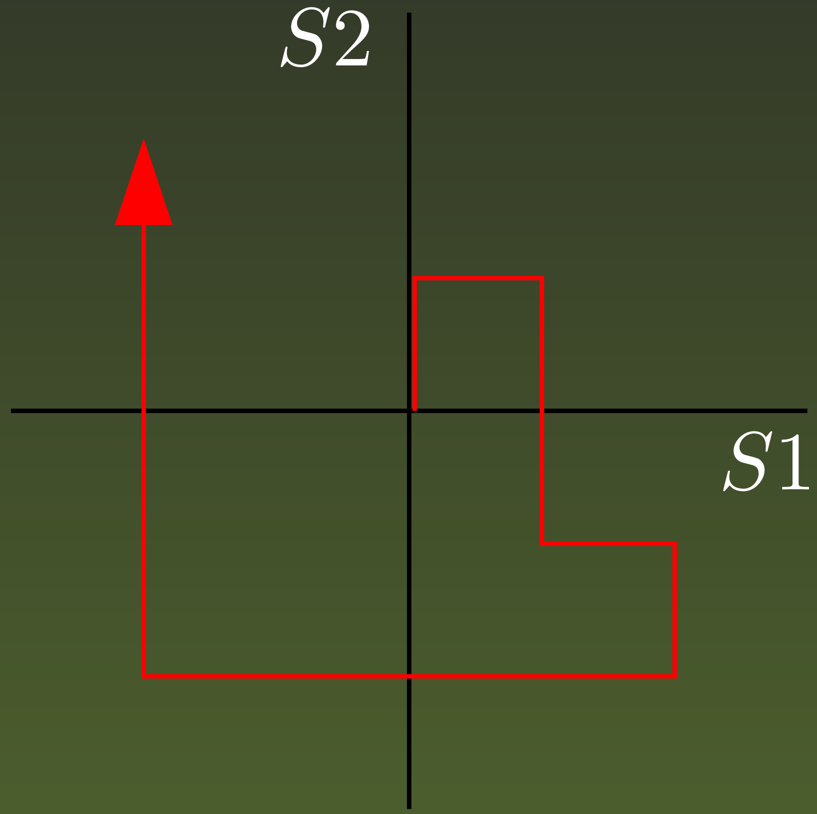


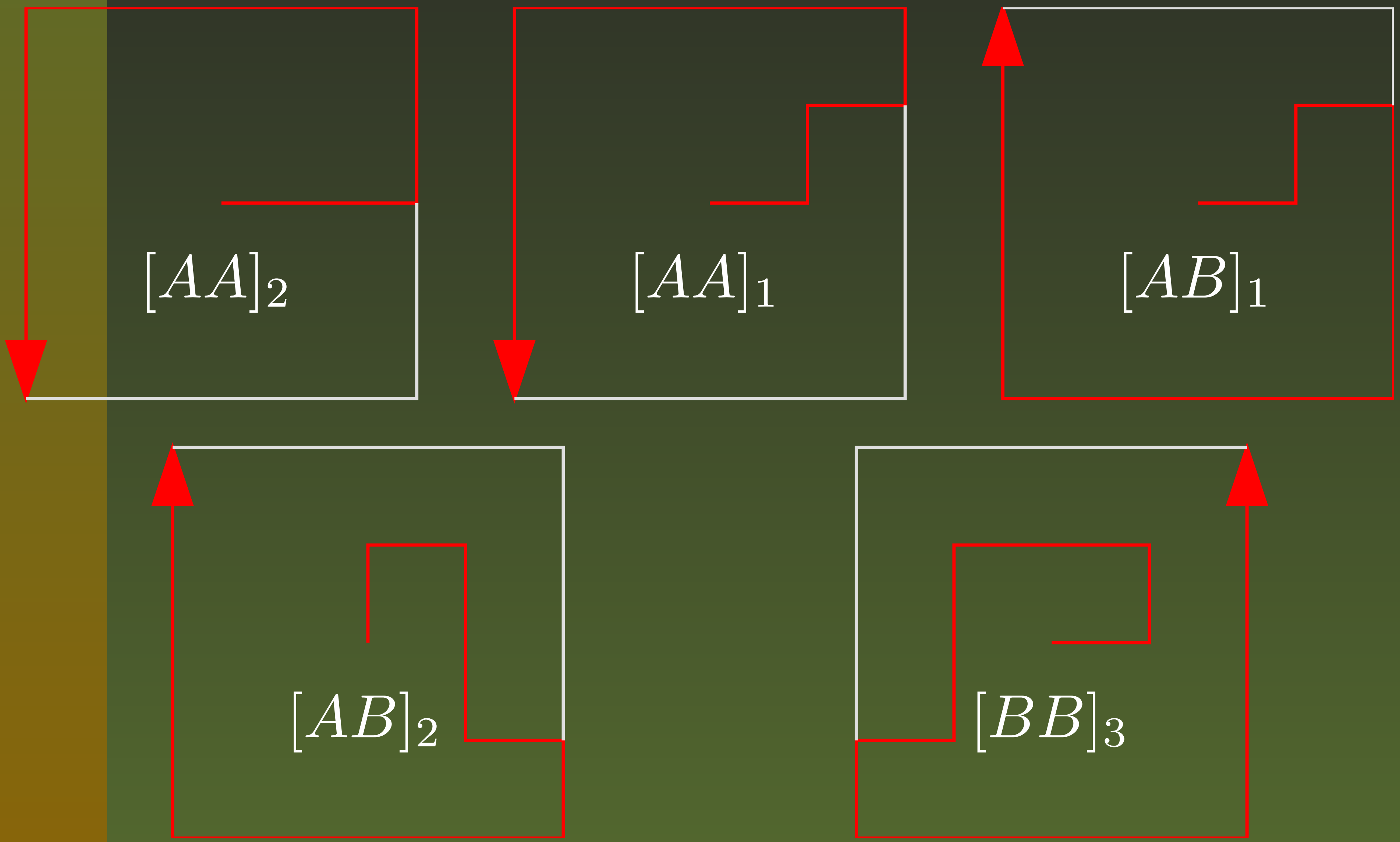
Expected time to find $= 5.425$.

Generalisation

- d is either 1 or 2, with respective probabilities p and $1 - p$
- Strategy A (efficient) is locally optimal for $p < \frac{2}{3}$
- Optimal strategies for different p ranges:

p range	local optimum	global optimum
$[0, 1/2]$	A (efficient)	$[AA]_2$
$[1/2, 3/5]$	A (efficient)	$[AA]_1$
$[3/5, 2/3]$	A (efficient)	$[AB]_1$
$[2/3, 8/11]$	B (greedy)	$[AB]_2$
$[8/11, 1]$	B (greedy)	$[BB]_3$

The 5 optimal strategies



Extensions

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- Two or more searchers in one region